

Introduction to Quantum Groups

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Abstract

We give an elementary introduction to the theory of algebraic and topological quantum groups (in the spirit of S. L. Woronowicz). In particular, we recall the basic facts from Hopf $(*)$ -algebra theory, theory of compact (matrix) quantum groups and the theory of their actions on compact quantum spaces. We also provide the most important examples, including the classification of quantum $SL(2)$ -groups, their real forms and quantum spheres. We also consider quantum $SL_q(N)$ -groups and quantum Lorentz groups.

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* Supported by Graduiertenkolleg “Mathematik im Bereich ihrer Wechselwirkung mit der Physik,” Dept. of Mathematics, Munich University and by Polish KBN grant No. 2 P301 020 07

1. Introduction and physical motivations

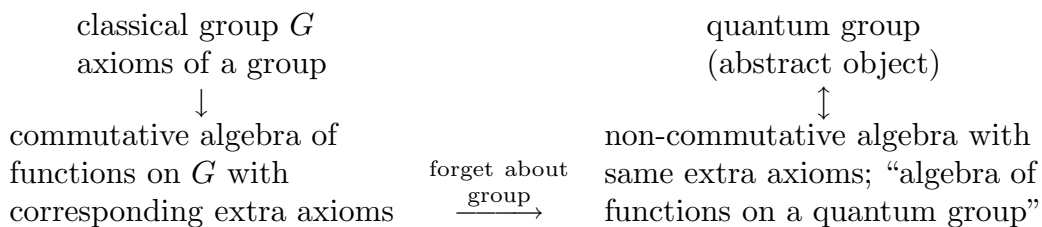
What are quantum groups?

Let G be a group in the usual sense, i. e. a set satisfying the group axioms, and k be a field. With this group one can associate a commutative, associative k -algebra of functions from G to k with pointwise algebra structure, i. e. for any two elements f and f' , for any scalar $\alpha \in k$, and $g \in G$ we have

$$(f + f')(g) := f(g) + f'(g), (\alpha f)(g) := \alpha f(g), (ff')(g) := f(g)f'(g)$$

If G is a topological group, usually only continuous functions are considered, and for an algebraic group the functions are normally polynomial functions. These algebras are called “algebras of functions on G .” These algebras inherit some extra structures and axioms for those structures from the group structure and its axioms on G . Locally compact groups can be reconstructed from this algebra.

Now the algebra is *deformed* or *quantized*, i. e. the algebra structure is changed so that the algebra is not commutative any more, but the extra structures and axioms for them remain the same. This algebra is called “algebra of functions on a quantum group”, where “quantum group” is just an abstract object “described” by the deformed algebra. This process can be summarized as follows:



There is a similar concept of “quantum spaces”: If G acts on a set X (e. g. a vector space), there is a corresponding so-called *coaction* of the commutative algebra of functions on G on the commutative algebra of functions on X satisfying certain axioms. The latter algebra can often be deformed/quantized into a non-commutative algebra, called the “algebra of functions on a quantum space” with a similar coaction. There are three ways of considering algebras of functions on a group and their deformations:

- (a) polynomial functions $\text{Poly}(G)$ (developed by Woronowicz and Drinfel’d),
- (b) continuous functions $C(G)$, if G is a topological group (developed by Woronowicz),
- (c) formal power series (developed by Drinfel’d).

Only the first two approaches will be dealt with in the sequel. They include representation theory, Peter-Weyl theory, Tannaka-Krein theory, and actions on quantum spaces.

There is a second approach to quantum groups. If G is a connected, simply connected Lie group, G can be reconstructed from the universal enveloping algebra $U(\mathfrak{g})$ of the corresponding Lie algebra \mathfrak{g} . The algebra $U(\mathfrak{g})$ again inherits some extra structures and axioms and can be deformed. The deformed universal enveloping algebra can be regarded as universal enveloping algebra corresponding to a quantum group. One can consider

- (d) the quantized universal enveloping algebra $U_q(\mathfrak{g})$ (developed by Jimbo),
- (e) formal power series (to be more precise, the ring of formal power series in \hbar over a free algebra, subject to certain relations which are the same as for $U(\mathfrak{g})$ in the case $\hbar = 0$. From this ring the algebra $U_q(\mathfrak{g})$ can be extracted. This approach has been developed by Drinfel'd).

This approach will not be used in the sequel.

Physical motivations

There are some physical motivations for quantum groups including

1. integrable models—handled with approach (e),
2. conformal field theory—handled with approach (e),
3. physical models based on quantized space-time—handled with approaches (a), (b), and (e).

The last motivation shall be explained in more detail. One of the main problems in Quantum Field Theory (QFT) is to join QFT and General Relativity Theory in a consistent way. It seems that in such a new theory it would be impossible to study the geometry of the space when very small volumes are considered. If you consider a cube in space, each vertex of it having Planck's length or less, and measure simultaneously the three coordinates x , y , and z of a particle in it, then the uncertainty of the measurement, i. e. the errors Δx , Δy , and Δz are very small, whence by Heisenberg's uncertainty relation the errors of the coordinates of the momentum are big and therefore the uncertainty of the energy ΔE is big, too. Since the energy is positive, the expected value $\langle E \rangle$ of the energy is big, and the smaller the cube the bigger the energy, which at a certain stage generates a black hole. Therefore the observation of the geometry of the space gives it a different geometry, which makes this observation useless (We have used here the arguments by Professor W. Nahm).

Quantum mechanics says that physical quantities such as momentum and position, which can be measured, correspond to self-adjoint operators on a Hilbert space. Its elements describe possible states of a physical system. When a quantity is measured, the state is projected onto an eigenvector of the operator, and the result of the measurement is the corresponding eigenvalue. Two quantities can be measured simultaneously if and only if the corresponding operators commute. In usual quantum mechanics the operators corresponding to the three coordinates of space commute and can be measured simultaneously,

which leads to the problem with the black hole. Thus it is reasonable to assume that the operators corresponding to the coordinates x , y , and z do not commute (whence they cannot be measured simultaneously). Hence the commutative algebra generated by the operators corresponding to x , y , and z , which is isomorphic to the algebra of polynomials on \mathbb{R}^3 , is replaced by a non-commutative algebra on a quantum space. In order to give sense to self-adjoint operators, this algebra should be a $*$ -algebra.

1.1. Definition: (a) A $*$ -algebra is a \mathbb{C} -algebra A equipped with an antilinear, antimultiplicative involution $*$: $A \rightarrow A$, i. e. for all $a, b \in A$ and $\lambda \in \mathbb{C}$ the following holds:

$$(a + b)^* = a^* + b^*, (\lambda a)^* = \bar{\lambda} a^*, (ab)^* = b^* a^*, (a^*)^* = a.$$

(b) Let A, B be $*$ -algebras. An algebra homomorphism $\phi: A \rightarrow B$ is called $*$ -homomorphism, if $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

Physical experiments should be comparable and reproducible, i. e. the same experiment performed at different places and times ought to give the same result. Therefore the theory should be invariant with respect to certain symmetry groups (containing translations in time and space). But the classical (symmetry) groups do not fit well to quantum spaces, so they have to be changed to quantum groups, too. (Example: The group $SO_3(\mathbb{R})$ of rotations in three-dimensional space acts on the sphere S^2 . When the algebra of functions on S^2 is properly deformed such that the algebra becomes non-commutative, then there is no reasonable coaction of the usual algebra of functions on $SO_3(\mathbb{R})$ any more. [P1, Remark 2])

There is another motivation—deformation of an existing physical theory may help to understand the theory in a better way. It can reveal why the theory works, what is a consequence, and what is just a coincidence.

Example [P4]: After looking at deformations of standard Dirac theory, the covariance of the Dirac equation can be seen more directly—on the level of groups rather than Lie algebras. For the wave vector Ψ there is the equation $\bar{\Psi} = \Psi^\dagger \gamma^0$, where γ^0 also appears in the Dirac equation. In the deformed theory there is $\bar{\Psi} = \Psi^\dagger A$ with $A \neq \gamma^0$ in general, so that $A = \gamma^0$ is just a coincidence, and the condition $A = \gamma^0$ is not really important for the theory.

In physics all symmetry groups are groups of matrices or can be described with groups of matrices, therefore the case of matrix groups is considered.

Acknowledgment

These lecture notes were written down by E. M. after the lectures by P. P. given at the Department of Mathematics, Munich University. The first author is very grateful to

Professor Dr. Hans-Jürgen Schneider for his warm hospitality at Munich University. We thank him for his useful remarks. An earlier version of these lectures was given by P. P. at Kyoto University in 1990–91. The first author would like to express his gratitude to Professor Huzihiro Araki for his kind hospitality and encouragement.

2. Polynomials on classical groups of matrices

Notations

In the sequel the base field of all vector spaces and algebras is the field \mathbb{C} of complex numbers. A *unital algebra* is an (associative) algebra with a unit element, and a *unital mapping* is a mapping between unital algebras which sends the unit element to the unit element.

Let \mathbb{N} , \mathbb{N}_0 and \mathbb{R} denote the sets of positive integers, non-negative integers and real numbers respectively and fix $M, N \in \mathbb{N}$. Let A be a unital algebra and let $M_{M \times N}(A)$ be the vector space of $M \times N$ -matrices with entries in A . If $M = N$, $M_N(A) := M_{N \times N}(A)$ is a unital algebra. For each matrix $M \in M_{M \times N}(A)$ let M_{ij} be the entry at the i -th row and j -th column of M . Let B be another algebra, $\phi: A \rightarrow B$ a map and $M \in M_{M \times N}(A)$. Then $\phi(M)$ is shorthand for the matrix in $M_{M \times N}(B)$ with entries $\phi(M_{ij})$. The group $GL(N, \mathbb{C})$ of invertible $N \times N$ -matrices with complex entries is equipped with a topology inherited from the norm topology of the vector space $M_N(\mathbb{C}) \cong \mathbb{C}^{N^2}$. The neutral element of a group is denoted by e .

Let \mathbb{C}^N denote the space of row vectors and ${}^N\mathbb{C}$ the space of column vectors. Using matrix multiplication, \mathbb{C}^N can be regarded as dual space of ${}^N\mathbb{C}$. If $\{e_1, \dots, e_N\}$ is a basis of ${}^N\mathbb{C}$, then there is a dual basis $\{e'_1, \dots, e'_N\}$ of \mathbb{C}^N such that $e'_i e_j = \delta_{ij}$ for all $i, j \leq N$. In a similar way there are dual bases of the k -fold tensor products $({}^N\mathbb{C})^{\otimes k}$ and $(\mathbb{C}^N)^{\otimes k}$.

In the sequel the indices i, j, i', j', k denote positive integers less or equal to N .

Let $\mathbf{1}_N$ denote the identity matrix with N rows and columns or the identity endomorphism of \mathbb{C}^N or ${}^N\mathbb{C}$.

Functions on groups

Let G be an arbitrary subgroup of the group $GL(N, \mathbb{C})$. Let $Fun(G)$ be the algebra of complex valued functions on G . This algebra is unital with unit element $\underline{1}: G \rightarrow \mathbb{C}, g \mapsto 1$ and is a $*$ -algebra, where for all $f \in Fun(G)$ the function f^* is defined by $f^*(g) := \overline{f(g)}$ for all $g \in G$.

For all i and j , the coefficient functions

$$u_{ij}: G \rightarrow \mathbb{C}, g \mapsto g_{ij} \text{ and } u_{ij}^{-1}: G \rightarrow \mathbb{C}, g \mapsto (g^{-1})_{ij}$$

belong to $Fun(G)$. Then the matrices $u := (u_{ij})_{1 \leq i, j \leq N}$ and $u^{-1} := (u_{ij}^{-1})_{1 \leq i, j \leq N}$ belong to $M_N(Fun(G))$ and are inverses of each other in $M_N(Fun(G))$. This justifies the notation u^{-1} .

2.1. Definition: Let $Pol(G)$ be the subalgebra of $Fun(G)$ generated by the elements u_{ij} and u_{ij}^{-1} for all i and j .

Remark: This algebra is automatically unital because of the relation $\underline{1} = \sum_{k=1}^n u_{1k} u_{k1}^{-1}$. The algebra is called “algebra of holomorphic polynomials on G ”, too.

2.2. Lemma: *If $G \subset SL(N, \mathbb{C})$ then $\text{Pol}(G)$ is already generated by the elements u_{ij} .*

Proof: By the usual formula for the inverse of a matrix, $(g^{-1})_{ij} = (-1)^{i+j} \det \tilde{g}_{j,i} / \det(g)$ for all $g \in G$, where the $(N-1) \times (N-1)$ -matrix $\tilde{g}_{j,i}$ is obtained from g by deleting the j -th row and the i -th column. But $\det(g) = 1$, whence also u_{ij}^{-1} is a polynomial in the functions $u_{i'j'}$.

2.3. Definition: Let $\text{Poly}(G)$ be the $*$ -subalgebra of $\text{Fun}(G)$ generated by the elements u_{ij} and u_{ij}^{-1} .

Usually the algebra $\text{Poly}(G)$ is considerably bigger than $\text{Pol}(G)$.

2.4. Lemma: *If G is a compact subgroup of $GL(N, \mathbb{C})$, then $\text{Poly}(G)$ is generated by the elements u_{ij} as $*$ -subalgebra.*

Proof: The map $\phi: G \rightarrow \mathbb{R}^+$, $g \mapsto |\det(g)|$ is a group homomorphism from G into the multiplicative group of positive real numbers. Since ϕ is continuous and G is compact, the image of ϕ is a compact subgroup of \mathbb{R}^+ . But $\{1\}$ is the only compact subgroup of \mathbb{R}^+ , whence $\phi(g) = 1$ for all $g \in G$. Therefore

$$1 = \det(g) \overline{\det(g)} = \det(g) \det((\bar{g}_{ij})_{1 \leq i,j \leq N}).$$

Thus $\det(u)$ is invertible in $\text{Poly}(G)$ with inverse $\det((u_{ij}^*)_{1 \leq i,j \leq N})$, whence the elements u_{ij}^{-1} can be expressed by the $u_{i'j'}$ and $u_{i'j'}^*$.

2.5. Remark: Let I be an index set and let G be a subgroup of $\prod_{\alpha \in I} GL(N_\alpha, \mathbb{C})$. Each element g of this group can be written as $g = (g_\alpha)_{\alpha \in I}$ with $g_\alpha \in GL(N_\alpha, \mathbb{C})$ for all $\alpha \in I$ and define

$$u_{ij}^\alpha, (u^\alpha)^{-1}_{ij}: G \rightarrow \mathbb{C}, \quad u_{ij}^\alpha(g) := (g_\alpha)_{ij}, \quad (u^\alpha)^{-1}_{ij} := (g_\alpha^{-1})_{ij}$$

for all $g \in G$. The algebras $\text{Pol}(G)$ and $\text{Poly}(G)$ are generated by the elements u_{ij}^α and $(u^\alpha)^{-1}_{ij}$ as algebras or $*$ -algebras, respectively. This generalization covers all compact groups G , because the group homomorphism

$$G \rightarrow \prod_{\pi \in \widehat{G}} GL(\dim(\pi), \mathbb{C}), \quad g \mapsto (\pi(g))_{\pi \in \widehat{G}}$$

where \widehat{G} is the set of finite dimensional irreducible representations of G , is injective if G is compact (cf. Tannaka-Krein duality).

The multiplication, unit, and the inverse on G lead to the following extra structures on $\text{Fun}(G)$:

$$\begin{aligned}\Delta: \text{Fun}(G) &\rightarrow \text{Fun}(G \times G), & (\Delta f)(g, h) &:= f(gh) \text{ for all } g, h \in G \text{ (Comultiplication),} \\ \varepsilon: \text{Fun}(G) &\rightarrow \mathbb{C}, & \varepsilon(f) &:= f(e) \text{ (Counit),} \\ S: \text{Fun}(G) &\rightarrow \text{Fun}(G), & (Sf)(g) &:= f(g^{-1}) \text{ for all } g \in G \text{ (Antipode).}\end{aligned}$$

These maps are unital $*$ -homomorphisms. The (algebraic) tensor product $\text{Fun}(G) \otimes \text{Fun}(G)$ is the vector subspace of $\text{Fun}(G \times G)$ generated by elements $u \otimes v$, where $u, v \in \text{Fun}(G)$, by defining $(u \otimes v)(g, h) := u(g)v(h)$ for all $g, h \in G$. Equality only holds if G is finite.

The axioms for the group structure on G are reflected by certain axioms for the extra structures on $\text{Fun}(G)$. Let f be an element of $\text{Fun}(G)$ such that $\Delta(f) \in \text{Fun}(G) \otimes \text{Fun}(G)$. Since the multiplication in G is associative, we have

$$(\Delta \otimes \text{id})\Delta(f) = (\text{id} \otimes \Delta)\Delta(f). \quad (1)$$

The property of the neutral element, namely $ge = eg = g$ for all $g \in G$, leads to the equation

$$(\varepsilon \otimes \text{id})\Delta(f) = (\text{id} \otimes \varepsilon)\Delta(f) = f. \quad (2)$$

(Here the usual identification $\mathbb{C} \otimes V \cong V \otimes \mathbb{C} \cong V$ for all \mathbb{C} -vector spaces is used). Let the linear map $\mu: \text{Fun}(G) \otimes \text{Fun}(G) \rightarrow \text{Fun}(G)$, $f \otimes f' \rightarrow ff'$ be induced by the multiplication in $\text{Fun}(G)$. Then the properties $gg^{-1} = g^{-1}g = e$ of the inverse can be expressed as

$$\mu(S \otimes \text{id})\Delta(f) = \mu(\text{id} \otimes S)\Delta(f) = \varepsilon(f)\underline{1}. \quad (3)$$

2.6. Definition: A unital algebra H is called *Hopf algebra*, if there are unital algebra homomorphisms $\Delta: H \rightarrow H \otimes H$ and $\varepsilon: H \rightarrow \mathbb{C}$ and a linear map $S: H \rightarrow H$ satisfying axioms (1)–(3) for all $f \in H$.

The following lemma gives examples of Hopf algebras and shows why $\text{Pol}(G)$ and the elements u_{ij}^{-1} are interesting.

2.7. Lemma: $\text{Pol}(G)$ is a Hopf algebra satisfying

$$\begin{aligned}\Delta u_{ij} &= \sum_{k=1}^N u_{ik} \otimes u_{kj}, & \Delta u_{ij}^{-1} &= \sum_{k=1}^N u_{kj}^{-1} \otimes u_{ik}^{-1}, \\ \varepsilon(u_{ij}) &= \varepsilon(u_{ij}^{-1}) = \delta_{i,j}, & S(u_{ij}) &= u_{ij}^{-1}, & S(u_{ij}^{-1}) &= u_{ij}.\end{aligned}$$

If G is finite, also $\text{Fun}(G)$ is a Hopf algebra.

Proof: For all $g, h \in G$,

$$\Delta u_{ij}(g, h) = u_{ij}(gh) = (gh)_{ij} = \sum_{k=1}^N g_{ik} h_{kj} = \sum_{k=1}^N u_{ik}(g) u_{kj}(h) = \sum_{k=1}^N (u_{ik} \otimes u_{kj})(g, h).$$

A similar computation yields the formula for $\Delta(u_{ij}^{-1})$. Therefore the image of $\text{Pol}(G)$ under Δ is contained in $\text{Pol}(G) \otimes \text{Pol}(G)$. The values of the counit can be computed: $\varepsilon(u_{ij}) = \varepsilon(u_{ij}^{-1}) = e_{ij} = \delta_{i,j}$. The equations for the antipode follow from $(S(u_{ij}))(g) = u_{ij}(g^{-1}) = u_{ij}^{-1}(g)$ for all $g \in G$. The Hopf algebra axioms are clearly satisfied, because $\text{Pol}(G)$ is a subalgebra of $\text{Fun}(G)$.

If G is finite, then $\text{Fun}(G)$ is a Hopf algebra because $\text{Fun}(G) \otimes \text{Fun}(G) = \text{Fun}(G \times G)$.

The following general theorem for Hopf algebras can be inferred from [A].

2.8. Theorem: *Let H be a Hopf algebra with unit element 1.*

- (a) *The maps ε and S are unique if Δ is fixed.*
- (b) *S is a unital antihomomorphism*
- (c) *If $\tau: H \otimes H \rightarrow H \otimes H$, $x \otimes y \mapsto y \otimes x$ denotes the flip automorphism, then*

$$\Delta S = \tau(S \otimes S)\Delta, \quad \varepsilon S = \varepsilon.$$

- (d) *Let $S': H \rightarrow H$ be a \mathbb{C} -linear map. Then the following are equivalent:*

- (i) $\mu(\text{id} \otimes S')\tau\Delta(f) = \mu(S' \otimes \text{id})\tau\Delta(f) = \varepsilon(f)1$ for all $f \in H$,
- (ii) $S \circ S' = S' \circ S = \text{id}$.

2.9. Remark: (a) In general, the antipode of a Hopf algebra is not invertible.

- (b) A map S' such as in part (d) of Theorem 2.8 is called *skew antipode*, and there is another Hopf algebra structure on H with comultiplication $\tau\Delta$, counit ε and antipode S' .
- (c) A motivation for the fact, that the counit, but not the antipode is an algebra homomorphism, if H is not commutative: Since Δ and the identity are algebra homomorphisms, there is no reason following from axiom (2) that ε should not be an algebra homomorphism. But the map μ in axiom (3) is an algebra homomorphism if and only if H is commutative. Therefore it should not be expected that S is an algebra homomorphism.

For all $f \in \text{Fun}(G)$ satisfying $\Delta(f) \in \text{Fun}(G) \otimes \text{Fun}(G)$, the following equation holds:

$$\begin{aligned} \Delta(f^*)(x, y) &= f^*(xy) = \overline{f(xy)} = \overline{\Delta f(x, y)} = \overline{\sum f_1(x) f_2(y)} = \\ &= \sum f_1^*(x) f_2^*(y) = (* \otimes *) \Delta(f)(x, y) \end{aligned}$$

for all $x, y \in G$. This motivates the following definition.

2.10. Definition: A unital algebra H is called a *Hopf $*$ -algebra*, if H is both a Hopf algebra and a $*$ -algebra such that $\Delta(f^*) = (* \otimes *)\Delta f$ for all $f \in H$.

From the definitions and Lemma 2.7 follows immediately

2.11. Lemma: $\text{Poly}(G)$ is a Hopf $*$ -algebra, and if G is finite, also $\text{Fun}(G)$ is a Hopf $*$ -algebra.

2.12. Proposition: Let H be a Hopf $*$ -algebra. Then

- (a) For all $x \in H$, $\varepsilon(x^*) = \overline{\varepsilon(x)}$, i. e. ε is a $*$ -homomorphism.
- (b) $S \circ * \circ S \circ * = \text{id}$, in particular, S is bijective.

Proof: (a) Since the map $H \rightarrow \mathbb{C}$, $x \mapsto \overline{\varepsilon(x^*)}$ satisfies the properties of the counit, both are equal by Theorem 2.8, part (a), whence the assertion follows.

- (b) The map $* \circ S \circ *$ satisfies all properties of the skew antipode. By Theorem 2.8, part (d) it is equal to it. This implies the two equivalent equalities $* \circ S \circ * \circ S = \text{id}_H = S \circ * \circ S \circ *$.

Elements of representation theory

Let H be a Hopf algebra.

2.13. Definition: Let k be a positive integer. A matrix $v \in M_k(H)$ is called *corepresentation*, if the entries satisfy the following relations for all indices a and b .

- (a) $\Delta v_{ab} = \sum_{c=1}^k v_{ac} \otimes v_{cb}$,
- (b) $\varepsilon(v_{ab}) = \delta_{a,b}$

The number $\dim v := k$ is called the *degree* of the corepresentation, and the elements v_{ab} are called the *matrix elements* of the corepresentation.

2.14. Remark: (a) Let v be a corepresentation of a Hopf algebra H . Then $S(v_{ab}) = (v^{-1})_{ab}$ for all indices a, b . Thus Condition (b) of Definition 2.13 can be equivalently replaced by invertibility of v (note that Condition (a) implies $\varepsilon(v)v = v$).

- (b) Let G be a classical group of matrices and H one of the Hopf algebras $\text{Pol}(G)$ or $\text{Poly}(G)$. Let

$$v: G \rightarrow M_k(\mathbb{C}), \quad g \mapsto (v_{ab}(g))_{1 \leq a, b \leq k}$$

be a map such that all functions v_{ab} are contained in H . Then $(v_{ab})_{1 \leq a, b \leq k}$ is a corepresentation if and only if v is a representation of G .

Proof: (a) This follows from the axioms for the antipode of a Hopf algebra.

- (b) For all $x, y \in G$ the following equations hold.

$$\begin{aligned} (\Delta v_{ab})(x, y) &= v_{ab}(xy) = (v(xy))_{ab}, \\ \left(\sum_{c=1}^k v_{ac} \otimes v_{cb} \right)(x, y) &= \sum_{c=1}^k v_{ac}(x) v_{cb}(y) = (v(x)v(y))_{ab}. \end{aligned}$$

Therefore condition (a) in Definition 2.13 is equivalent to $v(xy) = v(x)v(y)$. A computation of $\varepsilon(v_{ab})$ shows that condition (b) is equivalent to $v(e) = \mathbf{1}_k$.

Now fix a Hopf algebra H .

2.15. Definition: Let v and w be two corepresentations of H .

- (a) Then $v \oplus w$ and $v \otimes w$ are corepresentations of H , where $v \oplus w$ is a matrix with $\dim(v) + \dim(w)$ rows and columns given by

$$\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix},$$

and the matrix of $v \otimes w$ has $\dim(v)\dim(w)$ rows and columns and entries given by $(v \otimes w)_{ij,kl} := v_{ik}w_{jl}$, where the indices i, k take values between 1 and $\dim(v)$ and the indices j, l between 1 and $\dim(w)$.

- (b) A $\dim(w) \times \dim(v)$ matrix A over \mathbb{C} *intertwines* v with w , if $Av = wA$. Define $\text{Mor}(v, w)$ as vector space of intertwining matrices between v and w . The elements of $\text{Mor}(v, w)$ can be regarded as \mathbb{C} -linear maps from $\mathbb{C}^{\dim v}$ to $\mathbb{C}^{\dim w}$. The corepresentations v and w are said to be *equivalent* ($v \cong w$) if $\dim(v) = \dim(w)$ and there is an invertible element in $\text{Mor}(v, w)$.

2.16. Definition-Lemma: Let w be a corepresentation of dimension N and $V \subset {}^N\mathbb{C}$ a subspace of dimension l . Then the following are equivalent:

- (a) For each $\varrho \in \text{Hom}(H, \mathbb{C})$ the statement $\varrho(w)V \subseteq V$ holds.
- (b) There is a corepresentation v and a basis a_1, \dots, a_l of V such that for the $N \times l$ -matrix $A_l := (a_1 \cdots a_l)$ the equation $wA_l = A_lv$ holds. This is equivalent to the condition that A_l is an injective intertwiner of v with w .
- (c) There is a corepresentation v of dimension l and an invertible matrix A , the first l columns of which are a basis of V and such that

$$wA = A \begin{pmatrix} v & * \\ 0 & * \end{pmatrix}.$$

If one of the equivalent conditions holds, then V is called “ w -invariant subspace”, and the corepresentation v in part (b) and (c) is called “subcorepresentation of w ” and we write $v = w|_V$ (Note that v depends on the chosen basis of V).

Proof: (a) \Rightarrow ((b) \iff (c)). Let a_1, \dots, a_l be a basis of V and extend it to a basis a_1, \dots, a_N of ${}^N\mathbb{C}$. Then let A_l be the $N \times l$ -matrix $(a_1 \cdots a_l)$ and A be the $N \times N$ -matrix $(a_1 \cdots a_N)$. Then A is invertible and let $B := A^{-1}wA$. Let $\varrho \in \text{Hom}(H, \mathbb{C})$. Then $A^{-1}\varrho(w)A = \varrho(B)$. Now condition (a) means that there is a matrix $C_\varrho \in M_l(\mathbb{C})$ such that $\varrho(w)A_l = A_lC_\varrho$, whence $\varrho(B)$ looks like

$$\begin{pmatrix} \varrho(v) & * \\ 0 & * \end{pmatrix},$$

where v is the submatrix of B consisting of the first l rows and columns. Since this holds for all linear forms, there is the matrix equation

$$A^{-1}wA = \begin{pmatrix} v & * \\ 0 & * \end{pmatrix} \iff wA = A \begin{pmatrix} v & * \\ 0 & * \end{pmatrix}$$

or, equivalently, by restriction $wA_l = A_lv$.

(b) \Rightarrow (a). From (b) it follows for all $\varrho \in \text{Hom}(H, \mathbb{C})$ that $\varrho(w)A_l = \varrho(v)A_l$, which gives $\varrho(w)V \subseteq V$.

2.17. Definition: Let w a corepresentation.

- (a) w is said to be *irreducible* if $w \neq 0$ and there is no subcorepresentation v such that $0 < \dim(v) < \dim(w)$.
- (b) w is called *completely reducible* if w is equivalent to a direct sum of irreducible subcorepresentations.

2.18. Lemma: *The intersection of invariant subspaces is an invariant subspace.*

Proof: This follows directly from Definition-Lemma 2.16, part (a).

2.19. Lemma: *Let $A \in \text{Mor}(v, w)$. Then $\text{Ker}(A)$ is v -invariant and $\text{Im}(A)$ is w -invariant.*

Proof: Use Definition-Lemma 2.16, part (a). For each $\varrho \in \text{Hom}(H, \mathbb{C})$ the equation $A\varrho(v) = \varrho(w)A$ follows. If $x \in \text{Ker}(A)$ then $A\varrho(v)x = \varrho(w)Ax = 0$, whence $\varrho(v)x \in \text{Ker}(A)$ and the kernel is v -invariant. If $y \in \text{Im}(A)$, say $y = Az$, then $\varrho(w)y = \varrho(w)Az = A\varrho(v)z$ is in the image of A , too.

2.20. Lemma (Schur): *Let v, w be irreducible corepresentations. If v and w are not equivalent, then $\text{Mor}(v, w) = \{0\}$. If v is irreducible, then $\text{Mor}(v, v) = \mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is the identity.*

Proof: Let $A \in \text{Mor}(v, w) \setminus \{0\}$. Since v and w are irreducible, by Lemma 2.19, A must be injective and surjective, whence v and w are equivalent. Now let $w = v$ and λ be an eigenvalue of $A \in \text{Mor}(v, v)$. Then $A - \lambda\mathbf{1} \in \text{Mor}(v, v)$ is not injective and therefore vanishes.

2.21. Remark: There is a relationship between finite dimensional right comodules of H and corepresentations.

2.22. Theorem: *Let H be a Hopf algebra.*

- (a) *The matrix elements of corepresentations span H .*¹

¹ This result is related to the fact that each element of a Hopf algebra is contained in a finite dimensional subcoalgebra.

- (b) *The matrix elements of a set of non-equivalent irreducible corepresentations are linearly independent.*
- (c) *The following are equivalent:*
- (i) *There is a set T of non-equivalent irreducible corepresentations such that the matrix elements of them form a basis of H .*
 - (ii) *Each corepresentation is completely reducible.²*

Moreover if (i) holds then T contains all non-equivalent irreducible corepresentations.

Proof: (a) Let $x \in H$. Then there is a number $N \in \mathbb{N}$, linearly independent elements x_1, \dots, x_N and y_1, \dots, y_N in H such that $\Delta(x) = \sum_{j=1}^N x_j \otimes y_j$. By coassociativity, $\sum_{j=1}^N \Delta(x_j) \otimes y_j = \sum_{j=1}^N x_j \otimes \Delta(y_j)$, whence there are elements v_{ij} of H such that

$$\Delta(x_j) = \sum_{i=1}^N x_i \otimes v_{ij}$$

for all j . Using coassociativity and the properties for the counit, from these equations it follows that the elements v_{ij} are matrix elements of a corepresentation and $x_j = \sum_i \varepsilon(x_i) v_{ij}$ for all j . But then

$$x = \sum_{j=1}^N x_j \varepsilon(y_j) = \sum_{i,j=1}^N \varepsilon(y_j) \varepsilon(x_i) v_{ij}$$

is a linear combination of matrix elements.

- (b) Use the arguments in the proof of [W2, Proposition 4.7].
- (c) The conclusion (ii) \Rightarrow (i) is now obvious, because by (a), the Hopf algebra is spanned by matrix elements of irreducible corepresentations, which are linearly independent by (b). The conclusion (i) \Rightarrow (ii) is proved in [P2, Appendix]. The last remark follows from (b).

2.23. Proposition: *Let $\{v_\alpha \mid \alpha \in I\}$ and $\{v'_\beta \mid \beta \in J\}$ be sets of irreducible corepresentations such that*

$$\bigoplus_{\alpha \in I} v_\alpha \cong \bigoplus_{\beta \in J} v'_\beta.$$

Then the multiplicities of equivalence classes of irreducible corepresentations are the same on both sides.³

Proof: The set $\text{Mor}(\bigoplus v_\alpha, \bigoplus v'_\beta)$ can be computed using Schur's lemma (Lemma 2.20). But this set must contain an invertible element, since both direct sums are equivalent.

² In the language of Hopf algebras this means that H is cosemisimple.

³ cf. Krull-Remak-Schmidt theorem

- 2.24. Definition-Lemma:** (a) *Let w be a corepresentation of a Hopf algebra H . Then also the matrix w^c with matrix elements $w_{ij}^c := S(w_{ji})$ is a corepresentation, the contragradient corepresentation to w .*
- (b) *Let w be a corepresentation of a Hopf $*$ -algebra H . Then also the matrix \bar{w} with matrix elements $\bar{w}_{ij} := w_{ij}^*$ is a corepresentation. Define w^* to be the transpose of \bar{w} .*
- (c) *A corepresentation w of a Hopf $*$ -algebra is called unitary if $\bar{w} = w^c$ or equivalently $ww^* = w^*w = \mathbf{1}_{\dim w}$.*

Proof: (a) and (b) follow from the identities $\Delta \circ S = \tau(S \otimes S)\Delta$, $\varepsilon S = \varepsilon$, $\Delta \circ * = (* \otimes *)\Delta$.

3. Examples of quantum groups

Quantum $SL(2)$ -groups

The simplest Lie group over the complex numbers, which is interesting and important in physics, is $SL(2, \mathbb{C})$. We want to find quantum analogues of $\text{Pol}(SL(2, \mathbb{C}))$. The corepresentations of this Hopf algebra have the following properties:

- (1) The irreducible corepresentations are w^α , where $2\alpha \in \mathbb{N}_0$.
- (2) $\dim(w^\alpha) = 2\alpha + 1$ for all α ,
- (3) $w^\alpha \otimes w^\beta \cong w^{|\alpha-\beta|} \oplus w^{|\alpha-\beta|+1} \oplus \dots w^{\alpha+\beta}$ (Clebsch Gordan),
- (4) Each corepresentation is completely reducible, or equivalently, the matrix elements w_{ij}^α span the Hopf algebra.

Remark: The fundamental corepresentation is $w := w^{1/2}$ given by

$$g \mapsto (g_{ij})_{1 \leq i, j \leq 2}$$

for $g \in SL(2, \mathbb{C})$, and w^0 is the identity.

3.1. Definition: A quantum $SL(2)$ -group is a Hopf algebra satisfying the properties (1)–(4).

3.2. Theorem: Up to isomorphism there are the following quantum $SL(2)$ -groups \mathcal{H} . The Hopf algebra \mathcal{H} is generated by the matrix elements w_{ij} ($1 \leq i, j \leq 2$) of the fundamental corepresentation $w := w^{1/2}$ and relations

$$(w \otimes w)E = E, E'(w \otimes w) = E',$$

where the base field \mathbb{C} is canonically embedded into \mathcal{H} and there is the following extra relation between the row vector $E' \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and the column vector $E \in {}^2\mathbb{C} \otimes {}^2\mathbb{C}$: Let $\{e_1, e_2\}$ be a basis of ${}^2\mathbb{C}$ and $\{e'_1, e'_2\}$ be a dual basis of \mathbb{C}^2 . There is the following presentation:

$$E = \sum_{i,j=1}^2 E_{ij} e_i \otimes e_j, E' = \sum_{i,j=1}^2 E'_{ij} e'_i \otimes e'_j.$$

Then the 2×2 matrices with entries E_{ij} and E'_{ij} are inverses. There is a basis $\{e_1, e_2\}$ of ${}^2\mathbb{C}$ such that

$$E = e_1 \otimes e_2 - q e_2 \otimes e_1 \hat{=} \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \text{ or } E = e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_1 \hat{=} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

where $q \in \mathbb{C} \setminus \{0\}$ must not be a non-real root of unity. In the first case the quantum group is called the standard deformation $SL_q(2)$, in the second case it is called the non-standard deformation $SL_{t=1}(2)$. The non-standard deformation $SL_{t=1}(2)$ is not isomorphic to any of the standard deformations, and two standard deformations $SL_q(2)$ and $SL_{q'}(2)$ are isomorphic if and only if $q = q'$ or $qq' = 1$.

3.3. Remark: (a) There is a set of non-standard deformations $SL_t(2)$ indexed by a parameter $t \in \mathbb{C} \setminus \{0\}$ corresponding to the vector $E_t = e_1 \otimes e_2 - e_2 \otimes e_1 + te_1 \otimes e_1$, but they are all equivalent to the deformation for $t = 1$, because if the basis vector e_1 is replaced by $e'_1 = e_1 t$ then

$$tE_t = e'_1 \otimes e_2 - e_2 \otimes e'_1 + e'_1 \otimes e'_1 \triangleq E_1.$$

Since the relations remain the same when E is multiplied by a non-zero scalar, the Hopf algebras are isomorphic.

- (b) For $t \rightarrow 0$, the vector E_t tends to the vector for $q = 1$.
- (c) Parts of the proof of Theorem 3.2 can be found e. g. in [DV], [W4], [KP].

To prepare the proof of Theorem 3.2, some extra definitions and lemmas are useful.

3.4. Definition: Let q be a complex number. Then a *Hecke algebra* of degree n is a unital algebra generated by elements $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations

$$\begin{aligned} \sigma_k \sigma_{k+1} \sigma_k &= \sigma_{k+1} \sigma_k \sigma_{k+1} \text{ for } 1 \leq k \leq n-2, \\ (\sigma_k - 1)(\sigma_k + q^2 1) &= 0, \\ \sigma_k \sigma_l &= \sigma_l \sigma_k \text{ for } |k - l| \geq 2. \end{aligned}$$

From these relations follows an important property of Hecke algebras and quotients of them:

3.5. Definition-Lemma: Let \mathcal{A} be a Hecke algebra as in Definition 3.4. Let π be an element of the symmetric group Π_n of degree n , i. e. a permutation of the set $I := \{1, 2, \dots, n\}$. Then π can be written as the composition of transpositions t_j (where t_j interchanges the elements j and $j + 1$ of I). The minimal number of such transpositions is called length of π and is denoted by $l(\pi)$. Let $\pi = t_{k_1} \cdots t_{k_l}$ be a decomposition of π into a minimal number $l = l(\pi)$ of transpositions. Then $\sigma_{k_1} \cdots \sigma_{k_l}$ does not depend on the actual choice of transpositions as far as their number is minimal. Therefore $\sigma_\pi := \sigma_{k_1} \cdots \sigma_{k_l}$ is well-defined.

3.6. Definition-Lemma: Let \mathcal{A} be a Hecke algebra as in Definition 3.4. Then define the element

$$S_n := \sum_{\pi \in \Pi_n} q^{-2l(\pi)} \sigma_\pi \in \mathcal{A}.$$

This element satisfies the property $(\sigma_k - 1)S_n = 0$ for $1 \leq k \leq n - 1$.

Proof: Let k be an integer between 1 and $n - 1$. Let $\pi \in \Pi_n$ be a permutation such that $\pi(k) > \pi(k + 1)$ and let $\pi' := t_k \pi$. If $t_{k_1} \cdots t_{k_l}$ is a decomposition of π' into a minimal

number of transpositions then $t_k t_{k_1} \cdots t_{k_l} = t_k \pi'$ is a decomposition of π into a minimal number of transpositions and $l(\pi) = l(\pi') + 1$. Therefore for all k

$$\begin{aligned} S_n &= \left(\sum_{\substack{\pi \in \Pi_n \\ \pi(k) > \pi(k+1)}} q^{-2l(\pi)} \sigma_\pi + \sum_{\substack{\pi \in \Pi_n \\ \pi(k) < \pi(k+1)}} q^{-2l(\pi)} \sigma_\pi \right) = \\ &= (\sigma_k q^{-2} \sum_{\substack{\pi \in \Pi_n \\ \pi(k) < \pi(k+1)}} q^{-2l(\pi)} \sigma_\pi + \sum_{\substack{\pi \in \Pi_n \\ \pi(k) < \pi(k+1)}} q^{-2l(\pi)} \sigma_\pi) = (q^{-2} \sigma_k + 1) \sum_{\substack{\pi \in \Pi_n \\ \pi(k) < \pi(k+1)}} q^{-2l(\pi)} \sigma_\pi \end{aligned}$$

and hence

$$(\sigma_k - 1)S_n = \underbrace{(\sigma_k - 1)(1 + q^{-2}\sigma_k)}_{=0 \text{ (Hecke algebra)}} \sum_{\substack{\pi \in \Pi_n \\ \pi(k) < \pi(k+1)}} q^{-2l(\pi)} \sigma_\pi = 0.$$

3.7. Remark: The Hecke algebra is a generalization of the symmetric group, and for $q = 1$ the Hecke algebra relations are just the relations between the transpositions of the symmetric group. Let V be a vector space. The symmetric group acts on $V^{\otimes n}$ by permutations of the tensor factors. The operator σ_k corresponding to a transposition t_k has the eigenvalues 1 and -1. The intersection of the kernels of all $\sigma_k - 1$ or of the kernels of all $\sigma_k + 1$ are called “totally symmetric vectors” or “totally antisymmetric vectors”, respectively. When a Hecke algebra (or a quotient of it) acts on $V^{\otimes n}$, then the eigenvalues are 1 and $-q^2$ due to the second Hecke algebra relation. The intersection of the kernels of all $\sigma_k - 1$ or of the kernels of all $\sigma_k + q^2$ is called the space of “totally q -symmetric vectors” or “totally q -antisymmetric vectors”. The element S_n is called “symmetrization operator”, which is justified by Definition-Lemma 3.6, which also explains the factor $q^{-2l(\pi)}$ in the definition of S_n .

Proof of Theorem 3.2: Let $K = {}^2\mathbb{C}$ be the space of column vectors and $K' = \mathbb{C}^2$ the dual space of row vectors. w^0 has always the matrix element 1 (because $\Delta(1) = 1 \otimes 1$). Let w be the fundamental corepresentation $w^{1/2}$. Then $w \otimes w \cong w^0 \oplus w^1$, which is equivalent to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & w' \end{pmatrix},$$

where $w' \in M_3(\mathbb{C})$. This matrix has the column eigenvector $E = (1 \ 0 \ 0 \ 0)^T$ and the row eigenvector $E' = (1 \ 0 \ 0 \ 0)$. Therefore there are the relations

$$E'(w \otimes w) = E' = w^0 E', \quad (w \otimes w)E = E = Ew^0.$$

Thus the vectors E and E' , considered as 4×1 oder 1×4 matrices, intertwine $w \otimes w$ and w^0 . Moreover

$$E'E \neq 0. \tag{4}$$

Now $(E' \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes E)$ can be regarded as an intertwiner of w with w , because $w \cong w \otimes w^0 \cong w^0 \otimes w$ and

$$(E' \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes E)(w \otimes w^0) = (E' \otimes \mathbf{1}_2)(w \otimes w \otimes w)(\mathbf{1}_2 \otimes E) = (w^0 \otimes w)(E' \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes E).$$

Since w is irreducible, by Schur's Lemma (2.20) $(E' \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes E)$ is a multiple of the identity, say λ times the identity. Using the coordinate representation of E and E' with respect to a basis $\{e_i \otimes e_j \mid 1 \leq i, j \leq 2\}$ of $K \otimes K$ and the dual basis $\{e'_i \otimes e'_j \mid 1 \leq i, j \leq 2\}$ of $K' \otimes K'$,

$$E = \sum_{i,j} E_{ij} e_i \otimes e_j, \quad E' = \sum_{i,j} E'_{ij} e'_i \otimes e'_j,$$

this condition becomes

$$\sum_{k=1}^2 E'_{ik} E_{kj} = \lambda \delta_{ij}.$$

Therefore the matrices \widehat{E} with entries E_{ij} and \widehat{E}' with entries E'_{ij} satisfy

$$\widehat{E}' \widehat{E} = \lambda \mathbf{1}_2.$$

If $\lambda = 0$ then \widehat{E} must have rank 1, because if it has rank 2, then $E' = 0$ and if it has rank 0 then $E = 0$ in contradiction to (4). Hence $\widehat{E}_{ij} = x_i y_j$ for some $x_1, x_2, y_1, y_2 \in \mathbb{C}$ and E has the form $E = x \otimes y$, where $x = x_1 e_1 + x_2 e_2$, $y = y_1 e_1 + y_2 e_2$. From $(w \otimes w)E = E$ it follows that

$$wx \otimes wy = x \otimes y, \quad x \otimes wy = w^{-1}x \otimes y.$$

Both sides are in $K \otimes K \otimes A$. Applying $\phi \otimes \text{id}_K \otimes \text{id}_A$, where ϕ is a linear form on K such that $\phi(x) = 1$, we get

$$wy = y \otimes (\phi \otimes \text{id}_A)(w^{-1}x).$$

Therefore $\mathbb{C}y$ is an w -invariant subspace in contradiction to the fact that w is irreducible. Thus $\lambda \neq 0$, and by scaling of E' which does not change the relations, one gets $\widehat{E}' = \widehat{E}^{-1}$. The vector E in $K \otimes K$ can be written as sum of a symmetric tensor E^{sym} , i. e. an element of $K \otimes K$ which is invariant with respect to the flip automorphism τ of $K \otimes K$, mapping $x \otimes y$ to $y \otimes x$, and an antisymmetric tensor E^{asym} satisfying $\tau(E^{\text{asym}}) = -E^{\text{asym}}$, defined by $E^{\text{sym}} = \frac{1}{2}(E + \tau(E))$ and $E^{\text{asym}} = \frac{1}{2}(E - \tau(E))$. Symmetric tensors $\sum_{i,j} a_{ij} e_i \otimes e_j$ in $K \otimes K$, where $a_{ij} = a_{ji}$ for all i, j , can be identified with quadratic forms Q on K' , namely $Q(\sum_i v_i e'_i) = \sum_{i,j} a_{ij} v_i v_j$. In particular there are bases such that E^{sym} has one of the following presentations:

- (a) $E^{\text{sym}} = e_1 \otimes e_2 + e_2 \otimes e_1$ if Q has rank 2,
- (b) $E^{\text{sym}} = e_1 \otimes e_1$ if Q has rank 1,

(c) $E^{\text{sym}} = 0$ if Q has rank 0.

With respect to any basis $\{\tilde{e}_1, \tilde{e}_2\}$ of K an antisymmetric tensor E^{asym} is a scalar multiple of $\tilde{e}_1 \otimes \tilde{e}_2 - \tilde{e}_2 \otimes \tilde{e}_1$. Therefore E has one of the following presentations:

- (a) $E = (1 + c)e_1 \otimes e_2 + (1 - c)e_2 \otimes e_1$ with $c \in \mathbb{C}$. Since \hat{E} has rank 2, both coefficients must not vanish. Therefore E is a scalar multiple of $e_1 \otimes e_2 - qe_2 \otimes e_1$, where $q = \frac{c-1}{c+1}$ and $q \in \mathbb{C} \setminus \{0, 1\}$.
- (b) $E = e_1 \otimes e_1 + c(e_1 \otimes e_2 - e_2 \otimes e_1)$, where $c \in \mathbb{C} \setminus \{0\}$, because \hat{E} has rank 2. Therefore E is a scalar multiple of $e_1 \otimes e_2 - e_2 \otimes e_1 + te_1 \otimes e_1$, where $t = \frac{1}{c}$. According to Remark 3.3 this is equivalent to the vector for $SL_{t=1}(2)$. In this case let $q := 1$.
- (c) $E = c(e_1 \otimes e_2 - e_2 \otimes e_1)$, where $c \in \mathbb{C} \setminus \{0\}$. This is the case $q = 1$ which is not included in (a).

Now let the associative, unitary algebra \mathcal{H}_0 be generated by the elements $\alpha, \beta, \gamma, \delta$ subject to the relations $(v \otimes v)E = E, E'(v \otimes v) = E'$ for $v = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. There are uniquely determined comultiplication, counit and antipode such that this algebra becomes a Hopf algebra and v is a corepresentation (see Proposition 3.8 and Proposition 3.9 below).

Since the relations between the generators of \mathcal{H}_0 are satisfied in \mathcal{H} , there is a Hopf algebra map

$$\psi: \mathcal{H}_0 \rightarrow \mathcal{H}, v_{ij} \mapsto w_{ij}.$$

We shall study the corepresentation theory of \mathcal{H}_0 .

Consider the 4×4 matrix $\sigma := \mathbf{1}_4 + qE \cdot E'$ (where E and E' are again 4×1 and 1×4 matrices, respectively). Then σ is an element of the vector space $\text{Mor}(v \otimes v, v \otimes v)$. It satisfies the relations

$$(\sigma - \mathbf{1}_4)(\sigma + q^2 \mathbf{1}_4) = 0, \quad (\sigma \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes \sigma)(\sigma \otimes \mathbf{1}_2) = (\mathbf{1}_2 \otimes \sigma)(\sigma \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes \sigma).$$

Fix an integer $n \geq 2$ and define for integers k satisfying $0 < k < n$:

$$\sigma_k = \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{k-1} \otimes \sigma \otimes \underbrace{\mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2}_{n-k-1}.$$

These are operators on the n -fold tensor product $K^{\otimes n}$ and intertwine $v^{\otimes n}$ with $v^{\otimes n}$. They satisfy the *Hecke algebra relations* (cf. Definition 3.4).

Now define the operators σ_π as in Definition-Lemma 3.5 and the *symmetrization operator* as in Definition-Lemma 3.6:

$$S_n := \sum_{\pi \in \Pi_n} q^{-2l(\pi)} \sigma_\pi.$$

Due to Definition-Lemma 3.6 it takes values in

$$K^{n/2} := \{x \in K^{\otimes n} \mid \forall k: \sigma_k(x) = x\} = \bigcap_k \text{Ker}(\sigma_k - \mathbf{1}).$$

The dimension of the space $K^{n/2}$ is $n + 1$. (Proof: analyze relations on coordinates of elements of $K^{n/2}$ or see [W4]). The space $K^{n/2}$ is $v^{\otimes n}$ -invariant as intersection of the kernels of the intertwiners $\sigma_k - \mathbf{1}$ by Lemma 2.18 and Lemma 2.19 and a right comodule. Let $v^{n/2}$ denote the corresponding subcorepresentation of $v^{\otimes n}$ as in Definition-Lemma 2.16. Then $v^{n/2}$ is a corepresentation of dimension $n + 1$. By definition, $v = v^{1/2}$, v^0 is the trivial one-dimensional corepresentation. At this moment we assume that q is not a non-real root of unity. For all $s \in \frac{1}{2}\mathbb{N}_0$, one has

(A) v^k is irreducible for all $k \leq s + \frac{1}{2}$,

(B) $v^k \otimes v \cong v^{k+\frac{1}{2}} \oplus v^{k-\frac{1}{2}}$, where by definition $v^{-1/2} := 0$, for all $k \leq s$.

These statements will be proved by induction: The case $s = 0$ follows from the result on monomials below.

Suppose the statements are true for s replaced by $s - \frac{1}{2}$.

We want to decompose $v^s \otimes v$ and consider the map

$$\phi: K^{s-\frac{1}{2}} \rightarrow K^s \otimes K, x \mapsto (S_{2s} \otimes \mathbf{1}_2)(x \otimes E)$$

(antisymmetrization-symmetrization procedure). Note that $E \in K \otimes K$. The map is well defined due to the property of the symmetrization operator and intertwines $v^{\otimes(2s-1)} \cong v^{\otimes(2s-1)} \otimes v^0$ with $v^{\otimes 2s} \otimes v$. By inspection

$$(\sigma_{2s} - \mathbf{1})\phi(\underbrace{e_1 \otimes \cdots \otimes e_1}_{2s-1 \text{ factors}}) \neq 0$$

if q is not a non-real root of unity. Therefore $\phi(e_1 \otimes \cdots \otimes e_1) \notin K^{s+\frac{1}{2}}$ and $\text{Ker}(\phi) \neq K^{s-\frac{1}{2}}$. By induction hypothesis, there is no proper non-trivial v^{2s-1} -invariant subspace of $K^{s-\frac{1}{2}}$ and the kernel of ϕ is invariant by Lemma 2.19. Consequently ϕ is injective and $\text{Im}(\phi)$ corresponds to $v^{s-\frac{1}{2}}$. Moreover

$$\text{Im}(\phi) \cap K^{s+\frac{1}{2}} \neq \text{Im}(\phi).$$

Since $v^{s-\frac{1}{2}}$ is irreducible, there is no proper non-trivial $v^{s-\frac{1}{2}}$ -invariant subspace of $\text{Im}(\phi)$ and $\text{Im}(\phi) \cap K^{s+\frac{1}{2}} = \{0\}$. Thus

$$K^{s-\frac{1}{2}} \oplus K^{s+\frac{1}{2}} \cong \text{Im}(\phi) \oplus K^{s+\frac{1}{2}} \subset K^s \otimes K.$$

Equality follows by dimension arguments ($\dim K^t = 2t + 1$) and yields (B). By definition of the tensor product of representations, the monomials in $\alpha, \beta, \gamma, \delta$ of degree smaller or equal to $2s + 1$ are linear combinations of the matrix elements of $v^{\otimes(2s+1)}$. Using result (B) yields that they are linear combinations of matrix elements of $v^0, v^{\frac{1}{2}}, \dots, v^{s+\frac{1}{2}}$. The space

of these monomials has dimension $\sum_{k=1}^{2s+2} k^2$, as in the classical case for $\text{Pol}(SL(2, \mathbb{C}))$. This has been shown in [W4], [WZ2]. Since v^t has $(2t+1)^2$ matrix elements for all $t \in \frac{1}{2}\mathbb{N}_0$, the space spanned by the matrix elements of $v^0, v^{\frac{1}{2}}, \dots, v^{s+1/2}$ has this dimension if and only if all matrix elements are linearly independent. Hence (A) follows. Now it is easy to prove that \mathcal{H}_0 is a quantum $SL(2)$ -group.

The matrix $\psi(v^s)$ is a corepresentation of \mathcal{H} because ψ respects Δ and ε .

Then $\psi(v^s) \cong w^s$ for all $s \in \frac{1}{2}\mathbb{N}_0$. Proof by induction: The assertion is trivial for $s = 0$ and $s = \frac{1}{2}$. Suppose the statement is true for all non-negative half integers smaller than s . Then by part (B), $v^{s-\frac{1}{2}} \otimes v \cong v^{s-1} \oplus v^s$. Since ψ is an algebra homomorphism, by the definitions of direct sum and tensor product of corepresentations the following holds

$$\begin{aligned} w^{s-1} \oplus \psi(v^s) &\cong \psi(v^{s-1}) \oplus \psi(v^s) \cong \psi(v^{s-1} \oplus v^s) \cong \psi(v^{s-\frac{1}{2}} \otimes v) = \\ &= \psi(v^{s-\frac{1}{2}}) \otimes \psi(v) \cong w^{s-\frac{1}{2}} \otimes w \cong w^{s-1} \oplus w^s. \end{aligned}$$

Due to condition (4) for the quantum $SL(2)$ -group, the corepresentation $\psi(v^s)$ is completely reducible, whence by Proposition 2.23 $\psi(v^s) \cong w^s$. Thus ψ is an isomorphism and \mathcal{H} can be identified with \mathcal{H}_0 .

Now we consider the case when q is a non-real root of unity (see e. g. [KP]). Let q be a non-real root of unity of order N . Define

$$N_0 := \begin{cases} N & \text{if } N \text{ is odd,} \\ N/2 & \text{if } N \text{ is even.} \end{cases}$$

Then \mathcal{H}_0 has a corepresentation⁴

$$z = \begin{pmatrix} \alpha^{N_0} & \beta^{N_0} \\ \gamma^{N_0} & \delta^{N_0} \end{pmatrix}.$$

Then $v^k \otimes v \cong v^{k+\frac{1}{2}} \oplus v^{k-\frac{1}{2}}$ for $k < (N_0 - 1)/2$, v^k is irreducible for $k \leq \frac{1}{2}(N_0 - 1)$, and

$$v^{(N_0-1)/2} \otimes v \cong \begin{pmatrix} v^{\frac{1}{2}N_0-1} & * & * \\ 0 & z & * \\ 0 & 0 & v^{(N_0-1)/2} \end{pmatrix}.$$

It is possible to show $\psi(v^k) \cong w^k$ for $k \leq \frac{1}{2}(N_0 - 1)$ as before, but on the other hand

$$\begin{aligned} w^{\frac{1}{2}N_0-1} \oplus w^{\frac{1}{2}N_0} &\cong w^{\frac{1}{2}(N_0-1)} \otimes w \cong \psi(v^{\frac{1}{2}(N_0-1)} \otimes v) \cong \\ &\cong \begin{pmatrix} w^{\frac{1}{2}N_0-1} & * & * \\ 0 & \psi(z) & * \\ 0 & 0 & w^{\frac{1}{2}N_0-1} \end{pmatrix} \cong w^{\frac{1}{2}N_0-1} \oplus \psi(z) \oplus w^{\frac{1}{2}N_0-1}, \end{aligned}$$

⁴ cf. [T2, part 5.2]

because corepresentations in \mathcal{H} are completely reducible. But this is a contradiction to Proposition 2.23.

Let q_1 and q_2 be two values such that $SL_{q_1}(2) \cong SL_{q_2}(2)$ ($q_1, q_2 \in \mathbb{C} \setminus Y \cup \{t = 1\}$, where the subset Y contains 0 and all non-real roots of unity), i. e. that the Hopf algebras are isomorphic. Then the fundamental representation w_1 is mapped to w_2 , i. e. they are equivalent: $w_1 = Qw_2Q^{-1}$. Let E_1, E_2 be the corresponding eigenvectors. Then

$$\begin{aligned} (w_1 \otimes w_1)E_1 &= E_1, (Qw_2Q^{-1} \otimes Qw_2Q^{-1})E_1 = E_1 \Rightarrow \\ (w_2 \otimes w_2)((Q^{-1} \otimes Q^{-1})E_1) &= (Q^{-1} \otimes Q^{-1})E_1 = \lambda E_2 \end{aligned}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$, because the space of eigenvectors of $w_2 \otimes w_2$ for the eigenvalue 1 is one-dimensional. There are symmetric tensors $E_1^{\text{sym}}, E_2^{\text{sym}}$ and antisymmetric tensors $E_1^{\text{asym}}, E_2^{\text{asym}}$ such that $E_1 = E_1^{\text{sym}} + E_1^{\text{asym}}$ and $E_2 = E_2^{\text{sym}} + E_2^{\text{asym}}$. Therefore

$$E_1 = \lambda(Q \otimes Q)E_2 \Rightarrow E_1^{\text{sym}} = \lambda(Q \otimes Q)E_2^{\text{sym}}, E_1^{\text{asym}} = \lambda(Q \otimes Q)E_2^{\text{asym}} \quad (5)$$

and E_1^{sym} and E_2^{sym} have the same rank. If the rank is 0 or 1, it is the same deformation, and if the rank is 2, one can use (5) and the fact that the rank of $(Qe_1 \otimes Qe_2)$ is one, to get $q_1 = q_2$ or $q_1q_2 = 1$ (in the last case the isomorphism is given by $e_1 \leftrightarrow e_2$).

Quantum $SL(N)$ -groups

Let N be a positive integer greater than 1. The Hopf algebra \mathcal{H} of the group $SL(N, \mathbb{C})$ corresponds to the commutative unital algebra generated by the matrix elements w_{ij} for $1 \leq i, j \leq N$ of a fundamental corepresentation w subject to the relations

$$w^{\otimes N}E = E, E'w^{\otimes N} = E' \quad (6)$$

where E and E' are classical completely antisymmetric elements of $({}^N\mathbb{C})^{\otimes N}$ and $(\mathbb{C}^N)^{\otimes N}$ respectively, i. e. with respect to a basis $\{e_1, \dots, e_N\}$ of ${}^N\mathbb{C}$ and a dual basis $\{e'_1, \dots, e'_N\}$ of \mathbb{C}^N , they can be presented as

$$E = \sum_{\pi \in \Pi_n} (-1)^{l(\pi)} e_{\pi(1)} \otimes \dots \otimes e_{\pi(N)}, \quad E' = \sum_{\pi \in \Pi_n} (-1)^{l(\pi)} e'_{\pi(1)} \otimes \dots \otimes e'_{\pi(N)}. \quad (7)$$

Then the relations just mean (assuming commutativity) that the determinant of the matrix w is one. For $SL(2)$ this is just $e_1 \otimes e_2 - e_2 \otimes e_1$ and $e'_1 \otimes e'_2 - e'_2 \otimes e'_1$, which is changed to $e_1 \otimes e_2 - qe_2 \otimes e_1$ and up to a non-zero factor to $e'_1 \otimes e'_2 - qe'_2 \otimes e'_1$ in the standard deformation $SL_q(2)$. Therefore it is natural to define

$$E_q = \sum_{\pi \in \Pi_n} (-q)^{l(\pi)} e_{\pi(1)} \otimes \dots \otimes e_{\pi(N)}, \quad E'_q = \sum_{\pi \in \Pi_n} (-q)^{l(\pi)} e'_{\pi(1)} \otimes \dots \otimes e'_{\pi(N)} \quad (8)$$

and to consider the relations

$$w^{\otimes N} E_q = E_q, \quad E'_q w^{\otimes N} = E'_q. \quad (9)$$

For q not being a non-real root of unity they imply (cf. [W3])

$$w^{\otimes 2} \sigma = \sigma w^{\otimes 2} \quad (10)$$

where

$$\sigma(e_i \otimes e_j) := \begin{cases} qe_j \otimes e_i & \text{if } i < j, \\ qe_j \otimes e_i + (1 - q^2)e_i \otimes e_j & \text{if } i > j, \\ e_i \otimes e_i & \text{if } i = j, \end{cases}$$

for $i, j = 1, \dots, N$. Now $SL_q(N)$ is introduced as the unital algebra generated by w_{ij} for $1 \leq i, j \leq N$ subject to the relations (9), (10) (cf. [P2]). One can check that this definition coincides with the standard one (cf. [Dr], [R]).

The following proposition shows that all unital algebras with relations defined by intertwiners are bialgebras. If the intertwiners are chosen badly, the bialgebras can be small and uninteresting. For each matrix w and each $n \in \mathbb{N}$ define the matrix $w^{\otimes n}$ as for corepresentations in Definition 2.15 and let $w^{\otimes 0} := \mathbf{1}_1$.

3.8. Proposition: *Let \mathcal{H} be the universal unital algebra generated by elements w_{ij} for $1 \leq i, j \leq N$, which are the entries of a matrix w subject to relations*

$$E_m w^{\otimes s_m} = w^{\otimes t_m} E_m \quad (11)$$

for m in an index set I , $s_m, t_m \in \mathbb{N}_0$ and $E_m \in M_{N^{t_m} \times N^{s_m}}(\mathbb{C})$. Then there exist a unique comultiplication and counit such that \mathcal{H} is a bialgebra and w is a corepresentation of \mathcal{H} .

Proof: (a) Uniqueness: We must have $\Delta w_{ij} = \sum_{k=1}^N w_{ik} \otimes w_{kj}$ and $\varepsilon(w_{ij}) = \delta_{ij}$ for all i and j . Since Δ and ε are unital algebra homomorphisms, they are uniquely determined if they exist.

(b) Existence: Define $\hat{w}_{ij} := \sum_{k=1}^N w_{ik} \otimes_{\mathbb{C}} w_{kj} \in \mathcal{H} \otimes \mathcal{H}$ for all i and j . The matrix \hat{w} with entries \hat{w}_{ij} also satisfies the relations (11), because $\hat{w}^{\otimes n} = w^{\otimes n} \otimes_{\mathbb{C}} w^{\otimes n}$ follows from the rule $(a \otimes b)(c \otimes d) = (ac \otimes bd)$ and

$$\begin{aligned} E_m \hat{w}^{\otimes s_m} &= E_m (w^{\otimes s_m} \otimes_{\mathbb{C}} w^{\otimes s_m}) = w^{\otimes t_m} E_m \otimes_{\mathbb{C}} w^{\otimes s_m} = \\ &= w^{\otimes t_m} \otimes_{\mathbb{C}} E_m w^{\otimes s_m} = w^{\otimes t_m} \otimes_{\mathbb{C}} w^{\otimes t_m} E_m = \hat{w}^{\otimes t_m} E_m, \end{aligned}$$

because the entries of E_m are just complex numbers. Define $\tilde{w}_{ij} := \delta_{ij}$ for all i, j . Then the matrix \tilde{w} with entries \tilde{w}_{ij} satisfies the properties

$$E_m \tilde{w}^{\otimes s_m} = E_m, \quad \tilde{w}^{\otimes t_m} E_m = E_m,$$

whence it satisfies relations (11). Now the universality of \mathcal{H} gives the existence of unital homomorphisms Δ, ε such that $\Delta(w_{ij}) = \hat{w}_{ij}$ and $\varepsilon(w_{ij}) = \tilde{w}_{ij}$. It is enough to check Conditions (1) and (2) for bialgebras (cf. Definition 2.6) for elements $f = w_{ij}$ when they are obvious.

3.9. Proposition: *Let the conditions of Proposition 3.8 be satisfied. Let $\{e_1, \dots, e_N\}$ be a basis of ${}^N\mathbb{C}$ and $\{e'_1, \dots, e'_N\}$ be a dual basis of \mathbb{C}^N . Moreover assume that there exist positive integers s and t and elements $E \in \text{Mor}(\mathbf{1}_1, w^{\otimes t})$ and $E' \in \text{Mor}(w^{\otimes s}, \mathbf{1}_1)$ such that*

$$E = \sum_{k=1}^N e_k \otimes f_k, \quad E' = \sum_{k=1}^N f'_k \otimes e'_k$$

such that the elements $f_k \in ({}^N\mathbb{C})^{\otimes t-1}$ and $f'_k \in (\mathbb{C}^N)^{\otimes s-1}$ are linearly independent. Then the matrix w^{-1} exists and there is a uniquely determined antipode S such that the bialgebra \mathcal{H} is a Hopf algebra.

Proof: From the relation $w^{\otimes t}E = E$ it follows that

$$(w \otimes w^{\otimes(t-1)})E = E \Rightarrow \sum_{k=1}^N w e_k \otimes w^{\otimes(t-1)} f_k = \sum_{k=1}^N e_k \otimes f_k. \quad (12)$$

Since the elements f_k of $({}^N\mathbb{C})^{\otimes(t-1)}$ are linearly independent, there are elements g'_k of the dual space $(\mathbb{C}^N)^{\otimes(t-1)}$ such that $g'_i f_j = \delta_{ij}$. Apply $e'_i \otimes g'_j$ to Equation (12):

$$\sum_{k=1}^N e'_i w e_k \otimes g'_j w^{\otimes(t-1)} f_k = \sum_{k=1}^N e'_i e_k \otimes g'_j f_k = 1 \otimes g'_j f_i = \delta_{ij}.$$

Therefore the matrix G with entries $G_{kj} := g'_j w^{\otimes(t-1)} f_k$ is a right inverse to w . From the second condition it follows in a similar way that there is a left inverse of w . Thus w^{-1} exists. Finally, when to the relation

$$E_m w^{\otimes s_m} = w^{\otimes t_m} E_m,$$

$(w^{\otimes s_m})^{-1} = (w^{-1})^{\otimes \text{op } s_m}$ is applied to the right and $(w^{\otimes t_m})^{-1} = (w^{-1})^{\otimes \text{op } t_m}$ to the left (the tensor product “ \otimes^{op} ” is \otimes with respect to the algebra \mathcal{H}^{op} with opposite multiplication), then

$$(w^{-1})^{\otimes \text{op } t_m} E_m = E_m (w^{-1})^{\otimes \text{op } s_m}.$$

Therefore there is a unital algebra homomorphism $S: \mathcal{H} \rightarrow \mathcal{H}^{\text{op}}$ such that $S(w) = w^{-1}$. Equivalently, $S: \mathcal{H} \rightarrow \mathcal{H}$ is a unital antihomomorphism. It is enough to check Condition (3) for the antipode (cf. Definition 2.6) for $f = w_{ij}$ when it is obvious. Uniqueness of S follows from Theorem 2.8.

3.10. Remark: For the quantum $SL(N)$ group take $I = \{1, 2, 3\}$, $E_1 = E_q$, $t_1 = N$, $s_1 = 0$, $E_2 = E'_q$, $t_2 = 0$, $s_2 = N$, $E_3 = \sigma$, $t_3 = s_3 = 2$. Then the algebras $SL_q(N)$ are Hopf algebras.

3.11. Remark: For $0 < q \leq 1$ the corepresentation theory of $SL_q(N)$ is the same as for the classical $SL(N)$ (cf. [W3], [P2]). If q is transcendental, see [R], [H]. If $q \in \mathbb{C} \setminus \{0\}$ is not a non-real root of unity, see [PW]. There are deformations of the orthogonal and symplectic groups [RTF], [T1] (cf. [P2]).

4. *-Structures

In the classical theory there exist *-structures on $\text{Pol}(SL(2))$ which give the Hopf *-algebras $\text{Poly}(SU(2))$, $\text{Poly}(SU(1,1))$ and $\text{Poly}(SL(2, \mathbb{R}))$. We will classify the Hopf *-algebra structures on the quantum $SL(2)$ -groups \mathcal{H} described in Theorem 3.2. Firstly recall that \mathcal{H} is generated as an algebra by the matrix elements of a 2×2 matrix w subject to the relations

$$(w \otimes w)E = E, \quad E'(w \otimes w) = E'$$

or equivalently

$$\sum_{j,l} w_{ij} w_{kl} E_{jl} = E_{ik}, \quad \sum_{i,k} E'_{ik} w_{ij} w_{kl} = E'_{jl}. \quad (13)$$

4.1. Lemma: *Let ψ be an (anti-)linear comultiplicative algebra (anti-)automorphism of a quantum $SL(2)$ -group \mathcal{H} . Then*

- (a) *there exists a matrix $Q \in GL(2, \mathbb{C})$ such that $\psi(w) = QwQ^{-1}$.*
- (b) *If and only if the matrix $Q \in GL(2, \mathbb{C})$ satisfies the conditions*

$$(Q^{-1} \otimes Q^{-1})E = cE, \quad E'(Q \otimes Q) = c'E' \quad (14)$$

for some numbers $c, c' \in \mathbb{C} \setminus \{0\}$, there is a Hopf algebra automorphism ψ of \mathcal{H} such that $\psi(w) = QwQ^{-1}$. Moreover, all Hopf algebra automorphisms of \mathcal{H} can be described in this way.

- (c) *Let τ denote the linear twist (interchanging factors) and let \bar{E} and \bar{E}' denote the elements of ${}^2\mathbb{C} \otimes {}^2\mathbb{C}$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$ with conjugate complex coefficients with respect to the bases $e_i \otimes e_j$, $e'_i \otimes e'_j$. Then if and only if the matrix $Q \in GL(2, \mathbb{C})$ satisfies the conditions*

$$(Q^{-1} \otimes Q^{-1})\tau\bar{E} = cE, \quad \bar{E}'\tau(Q \otimes Q) = c'E' \quad (15)$$

for some $c, c' \in \mathbb{C} \setminus \{0\}$, there is an antilinear, comultiplicative, algebra antiautomorphism ψ of \mathcal{H} such that $\psi(w) = QwQ^{-1}$.

- (d) *Let the antilinear involutive comultiplicative algebra antiautomorphisms ψ , $\hat{\psi}$ and the corresponding matrices Q and \hat{Q} be defined as in (a). Then the Hopf algebra \mathcal{H} equipped with *-structures ψ and $\hat{\psi}$ gives isomorphic Hopf *-algebras if and only if $\hat{\psi}$ is equivalent to ψ up to a Hopf algebra automorphism ϕ (i. e. $\hat{\psi} = \phi\psi\phi^{-1}$) if and only if $\hat{Q} = c\bar{A}^{-1}QA$ where $c \in \mathbb{C} \setminus \{0\}$ and $A \in GL(2, \mathbb{C})$ corresponds to ϕ via (b).*

Proof: (a) Since ψ is comultiplicative, the matrix $\psi(w)$ is a corepresentation. The following conclusions follow from the fact that ψ is bijective: w is irreducible if and only if the matrix elements w_{ij} are linearly independent if and only if the matrix elements $\psi(w_{ij})$ are linearly independent if and only if $\psi(w)$ is irreducible. But there is only one

irreducible corepresentation of dimension 2 up to isomorphism, therefore there is a matrix $Q \in GL(2, \mathbb{C})$ such that

$$\psi(w) = QwQ^{-1}. \quad (16)$$

- (b) Since the trivial corepresentation appears in the direct sum decomposition $w \otimes w \cong w^1 \oplus w^0$ only once, by Lemma 2.20 the space of intertwiners in $\text{Mor}(w \otimes w, w^0)$ is one-dimensional. Thus Condition (14) is equivalent to the condition that $(Q^{-1} \otimes Q^{-1})E$ intertwines $w \otimes w$ with w^0 and $E'(Q \otimes Q)$ intertwines w^0 with $w \otimes w$:

$$\left. \begin{aligned} (w \otimes w)(Q^{-1} \otimes Q^{-1})E &= (Q^{-1} \otimes Q^{-1})E \iff (QwQ^{-1} \otimes QwQ^{-1})E = E \\ \text{and } E'(Q \otimes Q)(w \otimes w) &= E'(Q \otimes Q) \iff E'(QwQ^{-1} \otimes QwQ^{-1}) = E'. \end{aligned} \right\} \quad (17)$$

Let ψ be a Hopf algebra automorphism of \mathcal{H} . Then by part (a), there is a matrix $Q \in GL(2, \mathbb{C})$ such that $\psi(w) = QwQ^{-1}$. The automorphism ψ must map the relations between the generators of \mathcal{H} to relations in \mathcal{H} , therefore Equation (17) holds. Conversely, let Equation (17) be satisfied. Let \mathcal{F} be the free associative unital algebra generated by the matrix elements of w and let \mathcal{I} be the two-sided ideal generated by the relations (13). Then the map ψ can be defined as unital algebra homomorphism on \mathcal{F} such that $\psi(w) = QwQ^{-1}$. Equation (17) shows that ψ maps \mathcal{I} to \mathcal{I} , therefore it induces a unital algebra homomorphism on $\mathcal{H} = \mathcal{F}/\mathcal{I}$. Such ψ preserves the Hopf algebra structure of \mathcal{H} . Moreover, replacing Q by Q^{-1} (Equation (14) still holds for c^{-1} and $(c')^{-1}$) we get ψ^{-1} .

- (c) The proof is similar as in part (b). The only changes arise from the fact that ψ should be an antilinear algebra antiautomorphism instead of a linear algebra automorphism. Therefore, ψ applied to relations (13) yields

$$\sum_{j,l} \psi(w_{kl})\psi(w_{ij})\bar{E}_{jl} = \bar{E}_{ik}, \quad \sum_{i,k} \bar{E}'_{ik}\psi(w_{kl})\psi(w_{ij}) = \bar{E}'_{jl}$$

or shortly

$$\tau(\psi(w) \otimes \psi(w))\tau\bar{E} = \bar{E}, \quad \bar{E}'\tau(\psi(w) \otimes \psi(w))\tau = \bar{E}'.$$

Using $\tau^2 = \text{id}_{\mathcal{H}}$, we get the desired results.

- (d) $\phi\psi\phi^{-1}(w) = \phi\psi(A^{-1}wA) = \phi(\bar{A}^{-1}QwQ^{-1}\bar{A}) = \bar{A}^{-1}QA w A^{-1}Q^{-1}\bar{A}$, while $\hat{\psi}(w) = \hat{Q}w\hat{Q}^{-1}$. The left hand sides are equal if and only if $\hat{Q}^{-1}\bar{A}^{-1}QA \in \text{Mor}(w, w) = \mathbb{C}\mathbf{1}_2$.

Remark. It is easy to check that the second condition in (14) (and also the second condition in (15)) is redundant.

4.2. Theorem: All non-equivalent Hopf *-algebra structures on the quantum $SL_q(2)$ -groups \mathcal{H} are defined by $\bar{w} = QwQ^{-1}$, where

- (a) $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $|q| = 1$. Then $\bar{w} = w$. This algebra is called $\text{Poly}(SL_q(2, \mathbb{R}))$.
- (b) $Q = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$, $q \in \mathbb{R} \setminus \{0\}$. Then $w^*Bw = wBw^* = B$, for $B := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This algebra is called $\text{Poly}(SU_q(1, 1))$.
- (c) $Q = \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}$, $q \in \mathbb{R} \setminus \{0\}$. Then w is unitary. This algebra is called $\text{Poly}(SU_q(2))$.

The only equivalence among them is $\text{Poly}(SL_1(2, \mathbb{R})) \cong \text{Poly}(SU_1(1, 1))$.

For the non-standard deformation $SL_{t=1}(2)$ there is only one Hopf *-algebra structure (up to equivalence), namely for $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Except for (c), the above corepresentations w are not equivalent to unitary ones (The above examples were given in [W1], [RTF], [W4]).

Ideas of the proof: Since the map “*” is an antilinear comultiplicative algebra antiautomorphism, by Lemma 4.1, part (a) there is a matrix $Q \in GL(2, \mathbb{C})$ such that $\bar{w} = QwQ^{-1}$. By part (c) of Lemma 4.1, the map “*” can be an algebra antiautomorphism if and only if Q satisfies the condition

$$(Q^{-1} \otimes Q^{-1})\tau\bar{E} = cE$$

for some $c \in \mathbb{C} \setminus \{0\}$. The equation $*^2 = \text{id}_{\mathcal{H}}$ is equivalent to

$$\bar{Q}Q = d\mathbf{1}_2$$

with $d \in \mathbb{C} \setminus \{0\}$. Q is determined up to the equivalence relation as in Lemma 4.1, part (d). Consider the standard quantum deformations $SL_q(2)$, $q \neq 1$, first. From the relations (13) it follows that there are only the following characters (algebra homomorphisms) $\chi: \mathcal{H} \rightarrow \mathbb{C}$:

$$\chi_a(w) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ and in addition to that for } q = -1: \chi'_a(w) = \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix},$$

where $a \in \mathbb{C} \setminus \{0\}$ (Relations (13) are equivalent to

$$w_{11}w_{12} = qw_{12}w_{11}, \quad w_{11}w_{21} = qw_{21}w_{11}, \quad w_{12}w_{22} = qw_{22}w_{12},$$

$$w_{21}w_{22} = qw_{22}w_{21}, \quad w_{12}w_{21} = w_{21}w_{12},$$

$$w_{11}w_{22} - qw_{12}w_{21} = w_{22}w_{11} - q^{-1}w_{12}w_{21} = 1,$$

and the numbers $\chi(w_{ij})$ should satisfy the same relations).

Now the following trick can be used in order to compute all possible *-structures: If χ is a character, then also the map $\chi^\#: x \mapsto \overline{\chi(x^*)}$ is a character, because \mathbb{C} is commutative.

Then for any $a \in \mathbb{C} \setminus \{0\}$ there exists $b \in \mathbb{C} \setminus \{0\}$ such that $\chi_a^\# = \chi_b$ or (for $q = -1$) $\chi_a^\# = \chi'_b$. Applying both sides to w , we get that Q is a diagonal or antidiagonal matrix.

Similarly (use $\chi \mapsto \chi \circ \phi$), isomorphisms ϕ of Hopf algebras are given by diagonal or ($q = -1$) antidiagonal matrices. Then we use the other conditions for Q and part (d) of Lemma 4.1. For the non-standard deformation $SL_{t=1}(2)$ split E into E^{sym} and E^{asym} as in the proof of Theorem 3.2. Then consider Q with respect to both. For $q = 1$, equivalent Q 's can be regarded as matrices of the same antilinear mapping j such that $j^2 = d \cdot \text{id}$ (j is equivalent to kj for some $k \in \mathbb{C} \setminus \{0\}$). Then $d = 1$ corresponds to (a), (b) while $d = -1$ to (c).

4.3. Remark [RTF], [P2].

- (a) There exist the following *-structures on $SL_q(N)$:
 - (i) For $|q| = 1$ you can choose $\bar{w} = w$. The corresponding quantum group is called $SL_q(N, \mathbb{R})$.
 - (ii) If q is real then for $\varepsilon_1, \dots, \varepsilon_N \in \{\pm 1\}$ there are *-structures such that $w^* B w = w B w^* = B$, where B is a diagonal matrix with diagonal elements $\varepsilon_1, \dots, \varepsilon_N$. The corresponding quantum group is called $SU_q(N; \varepsilon_1, \dots, \varepsilon_N)$. For $\varepsilon_1 = \dots = \varepsilon_N = 1$ we get the quantum group $SU_q(N)$, in which w is a unitary corepresentation.
- (b) There are also *-structures on the orthogonal and symplectic quantum groups.

5. Compact Hopf \ast -algebras

In this chapter we follow [W2], [W3], [Ko]. Let \mathcal{A} be a Hopf \ast -algebra.

5.1. Definition: \mathcal{A} is called *compact* if there are unitary corepresentations such that their matrix elements generate \mathcal{A} as algebra.

Example: The fundamental corepresentation of $\text{Poly}(SU_q(N))$ is unitary and generates $\text{Pol}(SL_q(N))$ as algebra.

5.2. Lemma: Let \mathcal{A} be a compact Hopf \ast -algebra.

- (a) The matrix elements of unitary corepresentations span \mathcal{A} .
- (b) Let v be a unitary corepresentation. Then v is equivalent to a direct sum of irreducible unitary corepresentations.
- (c) The matrix elements of non-equivalent irreducible unitary corepresentations form a linear basis of \mathcal{A} .
- (d) Each irreducible corepresentation is equivalent to a unitary one.
- (e) Each corepresentation is completely reducible (into irreducible ones). Since the irreducible corepresentations are equivalent to unitary corepresentations, each corepresentation is equivalent to a unitary corepresentation.

Proof: (a) By definition, \mathcal{A} is spanned by matrix elements of tensor products of unitary corepresentations, but tensor products of unitary corepresentations are unitary.

- (b) Proof by induction with respect to the dimension d of corepresentations. If $d = 1$ or the corepresentation is irreducible, then there is nothing to do. Now assume that the corepresentation v is not irreducible. Then choose an orthonormal basis of an invariant proper subspace L and add some more orthonormal elements in order to get an orthonormal basis B of $\mathbb{C}^{\dim v}$. The transition from the standard basis to B is unitary and intertwines v with a unitary corepresentation

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} =: w,$$

where A, B, C are matrices of suitable size and with at least one entry. Since w is unitary, $\bar{w} = w^c$ or $S(w) = w^*$ or equivalently

$$\begin{pmatrix} S(A) & S(B) \\ 0 & S(C) \end{pmatrix} = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix}.$$

Therefore $B = 0$, moreover A and C are unitary and w a direct sum of them (notice that L^\perp is also invariant and $C = w|_{L^\perp}$). By induction hypothesis, the corepresentations A and C of dimensions less than d are direct sums of irreducible unitary corepresentations, whence w is a direct sum of irreducible unitary corepresentations.

- (c) This follows from (a), (b), and Theorem 2.22, part (b).
- (d) and (e) follow from Theorem 2.22, part (c).

Remark: All irreducible corepresentations can be obtained by decomposition of tensor products of those unitary corepresentations which generate \mathcal{A} as algebra (cf. Lemma 5.2, part (a)).

Peter-Weyl Theory and Haar measure

Let \mathcal{A} be a compact Hopf *-algebra. Let \mathcal{I} be an index set and let $\{u^\alpha \mid \alpha \in \mathcal{I}\}$ be a complete set of non-equivalent irreducible unitary corepresentations. Let $I := u^0$ be the one dimensional corepresentation. Then the elements u_{mn}^α form a basis of \mathcal{A} (Lemma 5.2, part (c)).

5.3. Definition: The *Haar measure* is a linear functional on \mathcal{A} defined by

$$h(u_{mn}^\alpha) = \delta_{\alpha,0}.$$

Since the u_{mn}^α are matrix elements of corepresentations, for all $x \in \mathcal{A}$ the Haar measure satisfies the equations

$$(h \otimes \text{id}_{\mathcal{A}})\Delta(x) = (\text{id}_{\mathcal{A}} \otimes h)\Delta(x) = h(x)1, h(1) = 1. \quad (18)$$

(By definition, also $h(S(x)) = h(x)$ holds for all $x \in \mathcal{A}$.)

In order to compute h on products, some preparation is necessary.

5.4. Lemma: For each $\alpha \in \mathcal{I}$ there is a strictly positive definite matrix F_α such that $(u^\alpha)^{cc} = F_\alpha u^\alpha F_\alpha^{-1}$.

Proof: For each $\alpha \in \mathcal{I}$, the matrix $\overline{u^\alpha}$ is also a corepresentation and equivalent to a unitary one, say u^β : $Q_\alpha \overline{u^\alpha} Q_\alpha^{-1} = u^\beta$. Then $\overline{u^\beta} = (u^\beta)^c$ and

$$(u^\alpha)^{cc} = (\overline{u^\alpha})^c = (Q_\alpha^{-1} u^\beta Q_\alpha)^c = Q_\alpha^T (u^\beta)^c (Q_\alpha^{-1})^T = Q_\alpha^T \overline{u^\beta} (Q_\alpha^{-1})^T = Q_\alpha^T \overline{Q_\alpha} u^\alpha \overline{Q_\alpha^{-1}} (Q_\alpha^{-1})^T$$

and therefore $(u^\alpha)^{cc} = F_\alpha u^\alpha F_\alpha^{-1}$ where $F_\alpha = Q_\alpha^T (Q_\alpha^T)^*$ is a strictly positive definite matrix.

Fix an irreducible corepresentation v and let $n := \dim(v)$. Since $S(v)$ is the inverse of v , there are intertwiners

$$AI = (v \otimes v^c)A, \quad B(v^c \otimes v) = IB,$$

where $A = \sum_{k=1}^n e_k \otimes e_k$ and $B = \sum_{k=1}^n e'_k \otimes e'_k$.

5.5. Lemma: *Let v and w be irreducible representations of dimensions n and m respectively. Then*

- (a) $\text{Mor}(v^c \otimes w, I) \cong \text{Mor}(w, v)$, $\text{Mor}(v^c \otimes v, I) = \mathbb{C}B$.
- (b) $\text{Mor}(I, w \otimes v^c) \cong \text{Mor}(v, w)$, $\text{Mor}(I, v \otimes v^c) = \mathbb{C}A$.

Proof: (a) If X intertwines $v^c \otimes w$ with I then $X(v^c \otimes w) = IX$ and

$$(\mathbf{1}_n \otimes X)(A \otimes \mathbf{1}_m)(I \otimes w) = (\mathbf{1}_n \otimes X)(v \otimes v^c \otimes w)(A \otimes \mathbf{1}_m) = (v \otimes I)(\mathbf{1}_n \otimes X)(A \otimes \mathbf{1}_m).$$

Since $I \otimes w \cong w$ and $v \otimes I \cong v$, $(\mathbf{1}_n \otimes X)(A \otimes \mathbf{1}_n)$ can be regarded as intertwiner of w and v . Conversely, let $Y \in \text{Mor}(w, v)$. Then $Yw = vY$ and

$$B(\mathbf{1}_n \otimes Y)(v^c \otimes w) = B(v^c \otimes v)(\mathbf{1}_n \otimes Y) = IB(\mathbf{1}_n \otimes Y).$$

Therefore $B(\mathbf{1}_n \otimes Y)$ intertwines $v^c \otimes w$ with I . The maps between $\text{Mor}(v^c \otimes w, I)$ and $\text{Mor}(w, v)$ are inverses of each other because $(\mathbf{1}_n \otimes B)(A \otimes \mathbf{1}_n) = (B \otimes \mathbf{1}_n)(\mathbf{1}_n \otimes A) = \mathbf{1}_n$. The second statement follows from the first with Schur's Lemma 2.20.

(b) is proved in a similar way.

Now the Haar measure is computed on certain products of basis elements:

5.6. Theorem: *The Haar measure satisfies the Peter-Weyl-Woronowicz relations:*

$$h(u_{mn}^\alpha u_{jl}^{\beta*}) = \delta_{\alpha,\beta} \frac{(F_\alpha)_{ln} \delta_{mj}}{\text{Tr}(F_\alpha)} \quad (19)$$

and

$$h(u_{jl}^{\beta*} u_{mn}^\alpha) = \delta_{\alpha,\beta} \frac{(F_\alpha^{-1})_{mj} \delta_{ln}}{\text{Tr}(F_\alpha^{-1})} \quad (20)$$

for all $\alpha, \beta \in \mathcal{I}$, $1 \leq m, n \leq \dim(u^\alpha)$, $1 \leq j, l \leq \dim(u^\beta)$.

Proof: Let w be any corepresentation (of dimension N). Application of $h \otimes \text{id}$ and $\text{id} \otimes h$ to Δw_{ij} yields together with Equation (18)

$$h(w)w = wh(w) = h(w)1.$$

This matrix equation means

$$\sum_{k=1}^N h(w)_{ik} w_{kj} = h(w)_{ij} 1 = \sum_{k=1}^N w_{ik} h(w)_{kj}$$

or equivalently that for each i the row vector with coordinates $h(w)_{ij}$ for $j = 1, \dots, N$ intertwines w with I and for each j the column vector with coordinates $h(w)_{ij}$ for $i =$

$1, \dots, N$ intertwines I with w . These facts will be applied to the sets $\text{Mor}(I, u^\alpha \otimes u^{\beta^c})$ and $\text{Mor}(u^{\alpha^{cc}} \otimes u^{\alpha^c}, I)$ for $\alpha, \beta \in \mathcal{I}$.

Therefore for $w = u^\alpha \otimes u^{\beta^c}$ and for fixed indices k, l , the element $(h(u_{ik}^\alpha u_{jl}^{\beta^c}))_{1 \leq i, j \leq N}$ is in $\text{Mor}(I, u^\alpha \otimes u^{\beta^c})$. By Lemma 5.5 it vanishes for $\alpha \neq \beta$ and is a multiple of A for $\alpha = \beta$. Thus there are numbers $\lambda_{kl}^\alpha \in \mathbb{C}$ such that

$$h(u_{ik}^\alpha u_{jl}^{\beta^c}) = \delta_{\alpha, \beta} \lambda_{kl}^\alpha \delta_{ij} \quad (21)$$

for all i, j, k, l . Similarly, for $w = u^{\alpha^{cc}} \otimes u^{\alpha^c}$ and for fixed indices i, j , the element $(h(u_{ik}^{\alpha^{cc}} u_{jl}^{\alpha^c}))_{1 \leq k, l \leq N}$ is in $\text{Mor}(u^{\alpha^{cc}} \otimes u^{\alpha^c}, I) = \mathbb{C}B$. Therefore there are numbers $\varrho_{ij}^\alpha \in \mathbb{C}$ such that

$$h(u_{ik}^{\alpha^{cc}} u_{jl}^{\alpha^c}) = \varrho_{ij}^\alpha \delta_{kl}. \quad (22)$$

But from Lemma 5.4, $u^\alpha = F_\alpha^{-1}(u^\alpha)^{cc}F_\alpha$, which yields by linearity and Equation (22) the equation

$$h(u_{mn}^\alpha u_{jl}^{\alpha^c}) = \sum_{i,k} (F_\alpha^{-1})_{mi} h(u_{ik}^{\alpha^{cc}} u_{jl}^{\alpha^c}) (F_\alpha)_{kn} = (F_\alpha)_{ln} \sum_i (F_\alpha^{-1})_{mi} \varrho_{ij}^\alpha.$$

Comparison with Equation (21) and $u_{jl}^{\beta^c} = (u_{jl}^\beta)^*$ yields

$$h(u_{mn}^\alpha u_{jl}^{\beta^c}) = c_\alpha \delta_{\alpha, \beta} (F_\alpha)_{ln} \delta_{mj}$$

for some $c_\alpha \in \mathbb{C}$. These constants can be evaluated using the unitarity of u^α :

$$1 = h(1) = \sum_n h(u_{mn}^\alpha u_{mn}^{\alpha^c}) = c_\alpha \sum_n (F_\alpha)_{nn} = c_\alpha \text{Tr}(F_\alpha).$$

The trace of F_α is positive because F_α is positive definite. This proves Equation (19). The other equation is proved in a similar way.

5.7. Remark: (a) Since the matrices F_α can be scaled by a positive number, we *normalize* them by the condition $\text{Tr}(F_\alpha) = \text{Tr}(F_\alpha^{-1})$. After normalization they are uniquely determined.

(b) Example: In the standard deformation $SU_q(2)$ for $q \in \mathbb{R} \setminus \{0\}$,

$$F_0 = (1), \quad F_{1/2} = \begin{pmatrix} |q|^{-1} & 0 \\ 0 & |q| \end{pmatrix}.$$

Proof: $w^{1/2} = w = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $S(w) = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}$. Then

$$w^{cc} = S^2(w) = \begin{pmatrix} \alpha & q^{-2}\beta \\ q^2\gamma & \delta \end{pmatrix} = F_{1/2} w F_{1/2}^{-1}$$

where $F_{1/2}$ is as desired. Note that the absolute value of q must be used, because the eigenvalues of a positive definite matrix must be positive.

5.8. Theorem (*Positivity of the Haar measure*)

For all $x \in \mathcal{A}$, $h(x^*x) \geq 0$, and equality only holds for $x = 0$.

Proof: Since \mathcal{A} has a basis $\{u_{mn}^\alpha \mid 1 \leq m, n \leq \dim(u^\alpha), \alpha \in I\}$, a general element a of \mathcal{A} can be written as

$$a = \sum_{m,n,\alpha} a_{mn}^\alpha u_{mn}^\alpha.$$

By the second Peter-Weyl-Woronowicz relation (20)

$$h(a^*a) = \sum_{\alpha,m,n,p} \frac{(a_{mp}^\alpha (F_\alpha^{-1})_{mn} \bar{a}_{np}^\alpha)}{\text{Tr}(F_\alpha)},$$

in which the sums $\sum_{m,n} a_{mp}^\alpha (F_\alpha^{-1})_{mn} \bar{a}_{np}^\alpha$ are strictly positive unless all coefficients a_{mp}^α for fixed α, p vanish, because the matrices F_α^{-1} are strictly positive definite for all α .

5.9. Corollary (*Scalar product*)

There is a scalar product on \mathcal{A} defined by $(a \mid b) := h(a^*b)$ for all $a, b \in \mathcal{A}$.

Proof: This inner product is antilinear in the first argument and linear in the second argument by definition and positive definite by Theorem 5.8.

5.10. Corollary (*Modular Homomorphism*)

There is a uniquely determined algebra automorphism σ of \mathcal{A} such that $h(ab) = h(b\sigma(a))$ for all $a, b \in \mathcal{A}$. It is defined on elements of the basis as

$$\sigma(u_{mn}^\alpha) = (F_\alpha u^\alpha F_\alpha)_{mn}.$$

Proof: Uniqueness: Let a be an element of \mathcal{A} and let $a', a'' \in \mathcal{A}$ such that for all $b \in \mathcal{A}$ the equation

$$h(ab) = h(ba') = h(ba'')$$

holds. Then $h(b(a' - a'')) = 0$ for all $b \in \mathcal{A}$, whence $a' = a''$ by Corollary 5.9.

Existence: From the second Peter-Weyl-Woronowicz relation it follows that

$$h(u_{jl}^{\alpha*} \sigma(u_{mn}^\beta)) = \frac{\delta_{\alpha,\beta} (F_\alpha)_{ln} \delta_{mj}}{\text{Tr}(F_\alpha)} = h(u_{mn}^\beta u_{jl}^{\alpha*}).$$

Therefore by linearity $h(ab) = h(b\sigma(a))$ for all $a, b \in \mathcal{A}$. Moreover $F_0 = (1)$ implies $\sigma(1) = 1$, and for all $a, b, c \in \mathcal{A}$,

$$h(a\sigma(bc)) = h(bca) = h(ca\sigma(b)) = h(a\sigma(b)\sigma(c)).$$

Therefore σ is a unital algebra homomorphism. Since F_α is invertible, also σ is invertible.

C^* -structure

For any Hilbert space H let $(\cdot | \cdot)_H$ denote the inner product and $B(H)$ the set of bounded linear operators on H . Then $B(H)$ is a $*$ -algebra. Let \mathcal{A} be a compact Hopf $*$ -algebra and consider the set

$$\Pi := \{\pi: \mathcal{A} \rightarrow B(H) \mid H \text{ Hilbert space, } \pi \text{ unital } *\text{-homomorphism}\}$$

(it is enough to consider some fixed H with $\dim(H) \geq \dim(\mathcal{A})$ as cardinal numbers, thus Π is actually a set).

Fix $\pi \in \Pi$ and let H be the corresponding Hilbert space. Let u^α be a unitary corepresentation of \mathcal{A} . Then $\pi(u^\alpha)$ is a unitary matrix in $M_{\dim u^\alpha}(B(H))$ and

$$\sum_m \pi(u_{mn}^\alpha)^* \pi(u_{mn}^\alpha) = 1$$

for all $n \leq \dim u^\alpha$. Therefore for all $x \in H$ and $k \leq \dim u^\alpha$

$$\begin{aligned} (x | x)_H &= \sum_m (\pi(u_{mn}^\alpha)^* \pi(u_{mn}^\alpha) x | x)_H = \\ &= \sum_m (\pi(u_{mn}^\alpha) x | \pi(u_{mn}^\alpha) x)_H \geq (\pi(u_{kn}^\alpha) x | \pi(u_{kn}^\alpha) x), \end{aligned}$$

whence the operator norm $\|\pi(u_{kn}^\alpha)\|$ is at most 1, and for each $a = \sum_{\alpha, m, n} a_{mn}^\alpha u_{mn}^\alpha \in \mathcal{A}$ there is the inequality

$$\|\pi(a)\| \leq \sum_{\alpha, m, n} |a_{mn}^\alpha| < \infty.$$

Therefore the following definition is possible:

5.11. Definition-Lemma: *There is a norm $\|\cdot\|_{C^*}$ on \mathcal{A} such that for all $a \in \mathcal{A}$,*

$$\|a\|_{C^*} = \sup_{\pi \in \Pi} \|\pi(a)\|.$$

Moreover this norm satisfies the equations $\|ab\|_{C^*} \leq \|a\|_{C^*} \|b\|_{C^*}$, $\|a^*\|_{C^*} = \|a\|_{C^*}$, $\|a^*a\|_{C^*} = \|a\|_{C^*}^2$ for all $a, b \in \mathcal{A}$.

Proof: The main problem is to show $\|a\|_{C^*} = 0 \Rightarrow a = 0$ for $a \in \mathcal{A}$. The inner product $(\cdot | \cdot)$ on \mathcal{A} induces a norm $\|\cdot\|_{(\cdot | \cdot)}$ (cf. Corollary 5.9). For each $a \in \mathcal{A}$ let $\pi_0(a)$ denote the operator of left multiplication by x on \mathcal{A} . Then for all $x \in \mathcal{A}$

$$\sum_m \|\pi_0(u_{mn}^\alpha)(x)\|_{(\cdot | \cdot)}^2 = h(x^* (\underbrace{\sum_m (u_{mn}^\alpha)^* u_{mn}^\alpha}_{=1}) x) = h(x^* x) = \|x\|_{(\cdot | \cdot)}^2,$$

whence the operator norm $\|\cdot\|'_{(\cdot, \cdot)}$ of $\pi_0(u_{mn}^\alpha)$ is at most 1. For all $a = \sum_{\alpha, m, n} a_{mn}^\alpha u_{mn}^\alpha \in \mathcal{A}$

$$\|\pi_0(a)\|'_{(\cdot, \cdot)} \leq \sum_{\alpha, m, n} |a_{mn}^\alpha|.$$

Therefore for each $a \in \mathcal{A}$ the operator $\pi_0(a)$ is bounded on \mathcal{A} and can be extended to the completion H of \mathcal{A} with respect to the norm $\|\cdot\|_{(\cdot, \cdot)}$ as a bounded linear operator $\bar{\pi}_0(a)$ with same operator norm $\|\bar{\pi}_0(a)\|'_{(\cdot, \cdot)} := \|\pi_0(a)\|'_{(\cdot, \cdot)}$. Therefore $\bar{\pi}_0 \in \Pi$, and

$$\|a\|_{C^*} = 0 \Rightarrow \|\pi_0(a)\|'_{(\cdot, \cdot)} = 0 \Rightarrow \|\pi_0(a)1\|_{(\cdot, \cdot)} = 0 \Rightarrow \|a\|_{(\cdot, \cdot)} = 0 \Rightarrow a = 0.$$

The other properties of this norm follow from the corresponding properties of the operator norms of the representations in Π .

5.12. Definition: Let A be the closure of \mathcal{A} with respect to the norm $\|\cdot\|_{C^*}$. Then A is a C^* -algebra by Definition-Lemma 5.11.

The following properties of C^* -algebras are useful:

5.13. Proposition: Let A be a C^* -algebra. Then

- (a) There is a Hilbert space H such that A can be embedded as closed $*$ -subalgebra into $B(H)$ [D, 2.6.1].
- (b) Let B be another C^* -algebra. Then each $*$ -homomorphism from A to B is continuous [D, 1.3.7].

The comultiplication of \mathcal{A} can be extended to a $*$ -homomorphism from A to $A \hat{\otimes} A$, where $A \hat{\otimes} A$ denotes the (topological) tensor product of C^* -algebras, defined as follows: Let H be a Hilbert space and let $\iota: A \rightarrow B(H)$ be an embedding of C^* -algebras. Then $A \hat{\otimes} A$ is identified with the closure of $(\iota \otimes \iota)(A \otimes A)$ in $B(H \hat{\otimes} H)$, where $H \hat{\otimes} H$ is the (topological) tensor product of Hilbert spaces. The C^* -algebra $A \hat{\otimes} A$ does not depend (up to isomorphisms) on the embedding ι [D, 2.12.15]. The map

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \hookrightarrow B(H \hat{\otimes} H)$$

is a $*$ -homomorphism called π_1 . Since $H \hat{\otimes} H$ is a Hilbert space, π_1 belongs to Π and can be extended to a $*$ -homomorphism on A . It is again called Δ .

5.14. Definition: A compact matrix quantum group is a pair (A, Δ) or shortly A where

- (a) A is a unital C^* -algebra generated by some elements $u_{ij} \in A$ for $1 \leq i, j \leq N$ and some positive integer N ,
- (b) $\Delta: A \rightarrow A \hat{\otimes} A$ is a unital $*$ -homomorphism such that $\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}$ for all i, j ,
- (c) the matrices u and \bar{u} are invertible.

5.15. Remark: (a) Let \mathcal{A} be a Hopf $*$ -algebra generated as unital algebra by matrix elements of *one* unitary corepresentation u or (equivalently) generated as unital $*$ -algebra by matrix elements of a corepresentation v such that v and \bar{v} are equivalent to unitary corepresentations. Then the C^* -algebra constructed as above is a compact matrix quantum group.

(b) For all positive integers N the compact Hopf $*$ -algebra of $SU_q(N)$ gives rise to a compact matrix quantum group.

(c) The general example of a compact matrix quantum group comes from C^* -algebras A as in (a) after dividing by closed two-sided ideals $I \subset \{x \in A: h(x^*x) = 0\}$ such that Δ induces a $*$ -homomorphism $A/I \rightarrow A/I \hat{\otimes} A/I$.

5.16. Theorem: *Let A be a compact matrix quantum group constructed as in Remark 5.15, part (c).*

(a) *Then $|h(x)| \leq \|x\|_{C^*}$ for all $x \in \mathcal{A}$, therefore h can be extended to a (positive) continuous functional on A , which will be denoted by h again.*

(b) *The algebra \mathcal{A} is embedded into A (because for all $x \in \mathcal{A} \setminus \{0\}$ the inequality $h(x^*x) > 0$ holds).*

(c) *Any corepresentation of A (in the sense $\Delta v_{ab} = \sum_c v_{ac} \otimes v_{cb}$, v^{-1} exists) has matrix elements in \mathcal{A} and thus \mathcal{A} can be recovered from A as the span of matrix elements of corepresentations.*

5.17. Remark: For $I_1 := \{x \in A: h(x^*x) = 0\}$ (it is a closed two-sided ideal due to [W2, p. 656]), h is faithful on A/I_1 (i. e. $h(x^*x) = 0 \Rightarrow x = 0$), while for $I_2 := \{0\}$, ε is continuous on $A/I_2 \cong A$. In the case of $SU_q(2)$, I_1 and I_2 coincide, cf. [P3, Remark 6].

The notion of compact matrix quantum groups generalizes that of algebras of continuous functions on compact groups of matrices. To be more precise: Let G be a compact group of matrices. Then there is a Haar measure μ on G . There is an inner product on $C(G)$ given by

$$(\chi, \psi) := \int_G \bar{\chi} \psi d\mu.$$

for $\chi, \psi \in C(G)$. The algebra $\text{Poly}(G)$ as in Definition 2.3 is a compact Hopf $*$ -algebra (cf. proof of Lemma 2.4). The inner product as above can also be expressed as $h(\chi^* \psi)$. Therefore the completion of $\text{Poly}(G)$ with respect to the norm $\|\cdot\|_{(\cdot, \cdot)}$ is the same as $L^2(G)$, and the completion of $\text{Poly}(G)$ with respect to the norm $\|\cdot\|_{C^*}$ is the same as $C(G)$. Here the comultiplication $\Delta: C(G) \rightarrow C(G) \hat{\otimes} C(G) \cong C(G \times G)$ is given by $\Delta(\chi)(g, h) = \chi(gh)$ for all $g, h \in G$ and $\chi \in C(G)$ (cf. Chapter 2). In the following, each compact topological space is by definition a Hausdorff space. There are one-to-one correspondences induced by

Gel'fand's theorem:

compact topological spaces $X \longleftrightarrow$ unital commutative C^* -algebras $C(X)$
continuous mappings $\lambda: X \rightarrow Y \longleftrightarrow$ unital $*$ -homomorphisms $\lambda^*: C(Y) \rightarrow C(X)$
cartesian product $X \times Y \longleftrightarrow$ topological tensor product $C(X) \hat{\otimes} C(Y)$
compact group of matrices $G \longleftrightarrow$ compact matrix quantum group $C(G)$
for commutative $\mathcal{A} = \text{Poly}(G)$

6. Actions on Quantum Spaces

Definition and spectral decomposition [P3, Section 1]

This chapter deals with a topological counterpart of right comodule algebras. Let V be a topological vector space and $Z \subset V$ a subset. Then $\langle Z \rangle$ denotes the closure of the linear span of the elements of Z in V .

6.1. Definition: Let (A, Δ) be a compact matrix quantum group and B a unital C^* -algebra. The unital $*$ -homomorphism $\Gamma: B \rightarrow B \hat{\otimes} A$ is called a *coaction* for A on B if

- (a) $(\Gamma \otimes \text{id}_A)\Gamma = (\text{id}_B \otimes \Delta)\Gamma$,
- (b) $B \otimes A = \langle (\text{id}_B \otimes y)\Gamma(x) \mid x \in B, y \in A \rangle$.

6.2. Remark: (a) Let G be a compact group of matrices, X a compact topological space and $X \times G \rightarrow X$, $(x, g) \mapsto xg$ for $x \in X$ and $g \in G$, an action. Then there is a coaction $\Gamma: C(X) \rightarrow C(X \times G)$ given by $\Gamma(\chi)(x, g) = \chi(xg)$ for all $\chi \in C(X)$, $g \in G$, $x \in X$. The properties $x(gh) = (xg)h$ and $xe = x$ for all $g, h \in G$, $x \in X$ correspond to Conditions (a) and (b) in Definition 6.1 respectively. Given a coaction as in Definition 6.1 for commutative A and B , the group action can be recovered by Gel'fand's theorem.

(b) Quantum analogues of left actions are considered in [P3, Remark 7].

6.3. Theorem: Let A be a compact matrix quantum group, B a unital C^* -algebra and Γ a coaction. Then there exists a maximal $*$ -subalgebra \mathcal{B} of B such that \mathcal{B} is dense in B and an \mathcal{A} right comodule algebra, i. e. for $\gamma := \Gamma|_{\mathcal{B}}$:

$$\gamma(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{A}, \quad (\gamma \otimes \text{id})\gamma = (\text{id} \otimes \Delta)\gamma, \quad (\text{id} \otimes \varepsilon)\gamma = \text{id}.$$

For each $\alpha \in \mathcal{I}$ there is a set I_α such that the algebra \mathcal{B} has a basis $e_{\alpha rk}$ for $\alpha \in \mathcal{I}$, $r \in I_\alpha$, $1 \leq k \leq \dim(u^\alpha)$ such that

$$\Gamma(e_{\alpha rk}) = \sum_s e_{\alpha rs} \otimes u_{sk}^\alpha.$$

Idea of proof (cf. [P3, Theorem 1.5]): From the Peter-Weyl-Woronowicz relation (20) it follows that there are elements $x_{sm}^\alpha \in \mathcal{A}$ which span \mathcal{A} such that the continuous linear functionals

$$\varrho_{sm}^\alpha: A \rightarrow \mathbb{C}, \quad x \mapsto h(x_{sm}^\alpha x)$$

satisfy $\varrho_{sm}^\alpha(u_{kr}^\beta) = \delta_{\alpha, \beta} \delta_{sk} \delta_{mr}$. Then the operators

$$E_{sm}^\alpha = (\text{id}_B \otimes \varrho_{sm}^\alpha)\Gamma: B \rightarrow B$$

have properties of matrix units. The traces $\sum_s E_{ss}^\alpha$ are projections onto subspaces $W_\alpha \subseteq B$ which contain all elements $x \in B$ such that

$$\Gamma(x) \subset B \otimes (\oplus_{ik} \mathbb{C} u_{ik}^\alpha).$$

Construction of the basis: For each $\alpha \in \mathcal{I}$ let $\{e_{\alpha r1} \mid r \in I_\alpha\}$ be a basis of the vector space $\text{Im}(E_{11}^\alpha)$ and $e_{\alpha rs} := E_{s1}^\alpha(e_{\alpha r1})$. Let \mathcal{B} denote the linear span of all elements $e_{\alpha rs}$. Then the closure of \mathcal{B} is

$$\begin{aligned} \langle E_{sm}^\alpha(x) \mid x \in \mathcal{B}, \alpha, s, m \rangle &= \langle E_{sm}^\alpha(x) \mid x \in B, \alpha, s, m \rangle = \\ &= \langle (\text{id} \otimes h)(\text{id} \otimes x_{sm}^\alpha) \Gamma(x) \mid x \in B, \alpha, s, m \rangle = \\ &= \langle (\text{id} \otimes h) \langle (\text{id} \otimes y) \Gamma(x) \mid y \in A, x \in B \rangle \rangle = \\ &= \langle (\text{id} \otimes h)(B \widehat{\otimes} A) \rangle = B. \end{aligned}$$

6.4. Definition: Let a compact matrix quantum group A coact by Γ on a quantum space B .

- (a) For each $\alpha \in \mathcal{I}$, the number c_α denotes the cardinality of I_α as in Theorem 6.3 and is called “multiplicity of u^α in the spectrum of Γ ”.
- (b) For each $\alpha \in \mathcal{I}$ let W_α be the linear span of the elements $e_{\alpha rs}$ as in Theorem 6.3.

Quantum spheres [P1]

Since the quantum groups $SU_q(2)$ and $SU_{1/q}(2)$ are isomorphic by Theorem 3.2, we can restrict ourselves to the case $q \in [-1, 1] \setminus \{0\}$. For the quantum $SU(2)$ groups, \mathcal{I} is the set of non-negative half integers and $u^k = w^k$ for $k \in \mathcal{I}$. We want to classify coactions Γ of $SU_q(2)$ such that

$$(1) \quad c_k = \begin{cases} 1 & \text{if } k \in \mathbb{N}_0 \\ 0 & \text{if } k \in \mathbb{N}_0 + \frac{1}{2}, \end{cases}$$

(2) the subspaces W_0 and W_1 generate B as a C^* -algebra.

The pairs (B, Γ) are called “quantum spheres” (cf. the case $q = 1$ in Theorem 6.5 below). For convenience, the matrix elements of the unitary irreducible corepresentations of $SU_q(2)$ will be indexed by numbers in the index set

$$N_\alpha := \{-\alpha, -\alpha + 1, \dots, \alpha\}$$

instead of the index set $\{1, \dots, 2\alpha + 1\}$ for each $\alpha \in \frac{1}{2}\mathbb{N}_0$.

6.5. Theorem [P1]: *In the case $q = 1$ there is only one object $B = C(S^2)$ and the coaction is induced by the standard right action of $SU(2)$ on the sphere S^2 . Here $W_0 = \mathbb{C}1$ and $W_1 = \mathbb{C}x + \mathbb{C}y + \mathbb{C}z$. Then Condition (2) means that the coordinates x, y, z separate the points of S^2 by the Stone-Weierstrass theorem.*

In the case $q = -1$ there is only one object $B_{-1,0}$ with coaction $\Gamma_{-1,0}$.

In the case $-1 < q < 1$ and $q \neq 0$ there are—up to isomorphisms—the following quantum spaces B_{qc} for $c \in \mathbb{R}_0^+ \cup \{\infty\}$. The C^ -algebra B_{qc} is generated by the elements e_{-1}, e_0, e_1 of W_1 subject to the relations*

$$\begin{aligned} e_i^* &= e_{-i} \text{ for } i \in \{-1, 0, 1\}, \\ \left. \begin{aligned} (1+q^2)(e_{-1}e_1 + q^{-2}e_1e_{-1}) + e_0^2 &= \varrho 1, \\ e_0e_{-1} - q^2e_{-1}e_0 &= \lambda e_{-1} \\ (1+q^2)(e_{-1}e_1 - e_1e_{-1}) + (1-q^2)e_0^2 &= \lambda e_0, \\ e_1e_0 - q^2e_0e_1 &= \lambda e_1, \end{aligned} \right\} \end{aligned} \quad (23)$$

where

$$\lambda = \begin{cases} 1 - q^2 & \text{if } c \in \mathbb{R} \\ 0 & \text{if } c = \infty \end{cases} \quad \text{and } \varrho = \begin{cases} (1+q^2)^2 q^{-2} c + 1 & \text{if } c \in \mathbb{R} \\ (1+q^2)^2 q^{-2} & \text{if } c = \infty. \end{cases}$$

The coaction Γ_{qc} is given by

$$\Gamma(e_i) = \sum_{j=-1}^1 e_j \otimes u_{ji}^1$$

for $i \in \{-1, 0, 1\}$. Here we choose a non-unitary form

$$u^1 = \begin{pmatrix} \delta^2 & -(q^2 + 1)\delta\gamma & -q\gamma^2 \\ -q^{-1}\beta\delta & 1 + (q + q^{-1})\beta\gamma & \alpha\gamma \\ -q^{-1}\beta^2 & (q + q^{-1})\beta\alpha & \alpha^2 \end{pmatrix}.$$

Ideas of proof: Due to Theorem 6.3 and Condition (1), the algebra \mathcal{B} has the linear basis $\{e_{\alpha k} \mid \alpha \in \mathbb{N}_0, k \in N_\alpha\}$ such that

$$\Gamma(e_{\alpha k}) = \sum_{s \in N_\alpha} e_{\alpha s} \otimes u_{sk}^\alpha \text{ for } \alpha \in \mathbb{N}_0, k \in N_\alpha.$$

Therefore the $e_{\alpha k}$'s are analogues of spherical harmonics. One has $(u_{lk}^1)^* = u_{-l, -k}^1$. Then

$$\Gamma(e_{-k}^*) = \sum_l e_{-l}^* \otimes (u_{-l, -k}^1)^* = \sum_l e_{-l}^* \otimes u_{lk}^1.$$

From the irreducibility of u^1 it follows that there is a constant c such that $e_{-k}^* = ce_k$ for all k . Moreover the modulus of c is one because of $e_k = (e_k^*)^* = (ce_{-k})^* = c\bar{c}e_k$. Thus it

is possible to achieve $c = 1$ by scaling the elements e_k with a suitable complex number of modulus one.

Now consider products of the generators: Because of the Clebsch-Gordan relation $u^1 \otimes u^1 \cong u^0 \oplus u^1 \oplus u^2$ there are injective intertwiners $G^\alpha \in \text{Mor}(u^\alpha, u^1 \otimes u^1)$ for $\alpha \in \{0, 1, 2\}$. From the equation

$$\Gamma(e_k e_l) = \sum_{m,r} e_m e_r \otimes u_{mk}^1 u_{rl}^1$$

it follows for the elements $\tilde{e}_{\alpha,t} := \sum_{k,l} e_k e_l G_{kl,t}^\alpha$:

$$\Gamma(\tilde{e}_{\alpha,t}) = \sum_{k,l,m,r} e_m e_r \otimes u_{mk}^1 u_{rl}^1 G_{kl,t}^\alpha = \sum_n \underbrace{\left(\sum_{m,r} e_m e_r G_{mr,n}^\alpha \right)}_{= \tilde{e}_{\alpha,n}} \otimes u_{nt}^\alpha.$$

Therefore the elements $\tilde{e}_{\alpha,t}$ satisfy the same relations for the coaction as the elements e_k . Since the corepresentations u^α are irreducible, there are constants $\lambda_\alpha \in \mathbb{C}$ such that $\tilde{e}_{\alpha,t} = \lambda_\alpha e_{\alpha,t}$. For $\alpha \in \{0, 1\}$ this gives relations for the generators:

$$\begin{aligned} \sum_{k,l} e_k e_l G_{rl,t}^1 &= \lambda e_t \quad (\text{here } \lambda = \lambda_1), \\ \sum_{k,l} e_k e_l G_{rl,0}^0 &= \varrho 1 \quad (\text{here } \varrho = \lambda_0). \end{aligned}$$

These are the relations (23) for the quantum spheres. Applying “*” to both sides, we obtain that λ and ϱ are real. There is still the freedom of scaling the e_k ’s by a non-zero real number. Consider the case $0 < |q| < 1$. If λ does not vanish, it can be scaled to the value $\lambda = 1 - q^2$. Then define c by

$$\varrho = (1 + q^2)^2 q^{-2} c + 1.$$

The existence of a faithful C^* -norm on \mathcal{B} implies that c is a non-negative number. It remains $\lambda = 0$, ϱ positive (\mathcal{B} is a C^* -algebra). Then ϱ can be scaled to the value $(1+q^2)^2 q^{-2}$. These (\mathcal{B}, Γ) ’s are indeed quantum spheres. No extra relation can be imposed, because then we would get a coaction for a quantum subspace. But $c_0 = 1$ means that the space is homogeneous (cf. [P3, Definition 1.8]), and from the facts that h is faithful (i. e. $h(x^*x) = 0 \Rightarrow x = 0$) and the counit is continuous (cf. Remark 5.17) it follows here that the homogeneous space corresponding to \mathcal{B} has no non-trivial homogeneous subspaces (this idea stands behind the proof in the paper [P1]).

The case $q = 1$ can be handled similarly, and the case $q = -1$ reduces to $q = 1$.

6.6. Remark: (a) If the first condition for the quantum spheres is weakened to $c_0 = c_1 = 1$, there are some more homogeneous spaces for $c \in \{c(2), c(3), \dots\}$, $0 < |q| < 1$, where

$$c(n) = -q^{2n}/(1 + q^{2n})^2 \text{ for all } n \in \mathbb{N}.$$

These objects satisfy the conditions

$$c_k = \begin{cases} 1 & \text{if } k = 0, 1, \dots, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

There exist analogues of these objects in the case $q = 1$. They correspond (cf. [P1]) to the adjoint action of $SU(2)$ on $U(\mathfrak{su}(2))$ taken in its n -dimensional irreducible $*$ -representation ($X^* = -X$ for $X \in \mathfrak{su}(2)$).

- (b) For $0 < |q| < 1$, $c \in \mathbb{R}_0^+ \cup \{\infty\} \cup \{c(2), c(3), \dots\}$ the quantum sphere $S_{qc}^2 = (B_{qc}, \Gamma_{qc})$ is a quotient space if and only if $c = 0$, embeddable (i. e. can be regarded as a non-zero C^* -subalgebra of A , where Γ is induced by the comultiplication) if and only if $c \in [0, \infty]$, and homogeneous for all considered c (for the compact groups of matrices these three notions coincide).
- (c) An algebraic version of Theorem 6.5 can be found in [S].

7. Quantum Lorentz groups (cf. [WZ2])

The algebra $\mathcal{A} = \text{Poly}(SL(2, \mathbb{C}))$ is called the algebra of polynomials on the Lorentz group. Its corepresentations have the following properties (cf. Chapter 3):

- (1) There are irreducible corepresentations w^α for $\alpha \in \frac{1}{2}\mathbb{N}_0$ such that all non-equivalent irreducible corepresentations are $w^\alpha \otimes \overline{w^\beta}$ for $\alpha, \beta \in \frac{1}{2}\mathbb{N}_0$.
- (2) $\dim(w^\alpha) = 2\alpha + 1$ for all α ,
- (3) $w^\alpha \otimes w^\beta \cong w^{|\alpha-\beta|} \oplus w^{|\alpha-\beta|+1} \oplus \dots \oplus w^{\alpha+\beta}$ (Clebsch Gordan),
- (4) Each corepresentation is completely reducible, or equivalently, the matrix elements $w_{ij}^\alpha (w_{kl}^\beta)^*$ give a basis of \mathcal{A} .
- (5) For all $\alpha, \beta \in \frac{1}{2}\mathbb{N}_0$ the corepresentations $w^\alpha \otimes \overline{w^\beta}$ and $\overline{w^\beta} \otimes w^\alpha$ are equivalent.

7.1. Definition: A quantum Lorentz group is a Hopf $*$ -algebra \mathcal{A} satisfying properties (1)–(5).

7.2. Theorem: Up to isomorphisms, all quantum Lorentz groups \mathcal{A} are given as follows: The Hopf $*$ -algebra \mathcal{A} is generated by the matrix elements w_{ij} ($1 \leq i, j \leq 2$) of the fundamental corepresentation $w := w^{1/2}$ and relations

- (i) $(w \otimes w)E = E$,
- (ii) $E'(w \otimes w) = E'$,
- (iii) $X(w \otimes \bar{w}) = (\bar{w} \otimes w)X$,

where the base field \mathbb{C} is canonically embedded into \mathcal{A} , the vectors $E' \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and $E \in {}^2\mathbb{C} \otimes {}^2\mathbb{C}$ are the same as in Theorem 3.2 and $X \in M_4(\mathbb{C})$ satisfies the properties:

- (iv) X is invertible,
- (v) there is a scalar factor $c \in \mathbb{C} \setminus \{0\}$ such that $\tau \bar{X} \tau = cX$,
- (vi) the intertwiners $\mathbf{1}_2 \otimes E$ and $(X \otimes \mathbf{1}_2)(\mathbf{1}_2 \otimes X)(E \otimes \mathbf{1}_2)$ in $\text{Mor}(\bar{w}, \bar{w} \otimes w \otimes w)$ are proportional (note that $\bar{w} \cong w^0 \otimes \bar{w} \cong \bar{w} \otimes w^0$).

Idea of proof: Necessity of relations: Restrict attention to the corepresentations w^α first. Their matrix elements give a basis of a quantum $SL(2)$ -group \mathcal{H} as in Theorem 3.2. This shows conditions (i) and (ii) and gives E and E' . From assertions (1) and (4) it follows that there is a linear isomorphism

$$\mathcal{A} \cong \mathcal{H} \cdot \mathcal{H}^* \cong \mathcal{H} \otimes \mathcal{H}^*, \quad w_{kl}^\alpha (w_{mn}^\beta)^* \mapsto w_{kl}^\alpha \cdot (w_{mn}^\beta)^* \mapsto w_{kl}^\alpha \otimes_{\mathbb{C}} (w_{mn}^\beta)^*,$$

where “ \cdot ” denotes multiplication. Assertion (5) for $\alpha = \beta = \frac{1}{2}$ shows that there is a bijective intertwiner $X \in \text{Mor}(w \otimes \bar{w}, \bar{w} \otimes w)$, which gives conditions (iii) and (iv). Apply the map “ $*$ ” to (iii) and use the formula $\overline{v \otimes w} = \tau(\bar{w} \otimes \bar{v})\tau$ as in the proof of Lemma 4.1, part (c):

$$\bar{X}(\overline{w \otimes \bar{w}}) = \overline{(\bar{w} \otimes w)}\bar{X} \Rightarrow \bar{X}\tau(w \otimes \bar{w})\tau = \tau(\bar{w} \otimes w)\tau\bar{X} \Rightarrow \tau\bar{X}\tau(w \otimes \bar{w}) = (\bar{w} \otimes w)\tau\bar{X}\tau.$$

Since $\bar{w} \otimes w$ and $w \otimes \bar{w}$ are irreducible, the intertwiners X and $\tau \bar{X} \tau$ must be proportional, which gives Condition (v). The last condition follows, because both $\mathbf{1}_2 \otimes \mathbf{1}_2$ and

$$X(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes E')(\mathbf{1}_2 \otimes X \otimes \mathbf{1}_2)(E \otimes \mathbf{1}_2 \otimes \mathbf{1}_2)$$

are elements of $\text{Mor}(\bar{w} \otimes w)$.

Existence: We set $\mathcal{A} := \mathcal{H} \otimes \mathcal{H}^*$ with \mathcal{H} as in Theorem 3.2 and laborously introduce the Hopf $*$ -algebra structure on \mathcal{A} by means of (iii)–(vi).

Sufficiency of relations: More relations would make the elements $w_{ij}^\alpha (w_{kl}^\beta)^*$ linearly dependent.

- 7.3. Remark:** (a) Possible matrices X have been found (up to isomorphisms of the corresponding Hopf $*$ -algebras) in [WZ2].
- (b) There is also a topological structure for two examples of \mathcal{A} ([PW1], [WZ1]) which uses the notion of affiliated elements [W5].
- (c) Quantum Poincaré groups arise by adding translations [PW2].
- (d) Quantum analogues of $\text{Poly}(SL(N, \mathbb{C}))$ were considered in [P2] (cf. [Z]).

8. References

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